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Arithmetization as a tool of discovery in Felix Klein's research program and epistemological writings

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Outline

1. Fortune (and misfortune) of Klein's "Erlanger Programm"
2. From projective metrical geometry to the group-theoretical view of geometry
3. Arithmetization in Klein's epistemological writings
4. Alternative philosophical views in the reception of Klein: Bertrand Russell and Ernst Cassirer
5. Concluding remarks on Klein and contemporary philosophy of mathematics

Fortune (and misfortune) of Klein's "Erlanger Programm"

- The received view, often advocated by mathematicians, is that the Erlangen Program was very influential
  - Birkhoff, G. and Bennett, M. K. "Felix Klein and his Erlanger Programm" (1988).
- Category theory as a generalization of Klein's Erlangen Program
  - Eilenberg, S. and Mac Lane, S. "A general theory of natural equivalences" (1945); cf. Marquis, J. P. *From a geometrical point of view: A study of the history and philosophy of category theory*, 2009.
- The delayed reception of the Erlangen Program
  - Hawkins, T. "The Erlanger Program of Felix Klein: Reflections on its place in the history of mathematics" (1984); Rowe, D. E. "The early geometrical works of Sophus Lie and Felix Klein" (1989).

Klein's work on projective geometry and group theory I

Klein, Felix. 1871a. "Notiz, betreffend den Zusammenhang der Liniengeometrie mit der Mechanik starrer Körper." *Mathematische Annalen* 4.

___. 1871b. "Über die sogenannte Nicht-Euklidische Geometrie." *Mathematische Annalen* 4, 573-625. Part 2, 1873.

___. 1872. "Vergleichende Betrachtungen über neuere geometrische Forschungen." Erlangen: Deichert. Repr. *Mathematische Annalen* 43 (1893)... Or the "Erlangen Program."

Lie, Sophus. 1888-1893. *Theorie der Transformationsgruppen*, 3 Bde., Leipzig.

Klein, Felix.1890. "Zur Nicht-Euklidischen Geometrie." *Mathematische Annalen* 37.

___. 1910. "Über die geometrischen Grundlagen der Lorentzgruppe." *Physikalische Zeitschrift* 12 (1911): 17-27.

Klein’s work on projective geometry and group theory II

In the same years [1890s] I was able to cultivate my interests in mechanics and mathematical physics, as I have been intending to do since the beginning of my studies in mathematics. The first physical investigations in the theory of relativity emerged a few years later, and rapidly attracted general attention. I suddenly recognized that my classification of 1872 included even these investigations and provided the simplest way to clarify the newest physical (or even philosophical) ideas from a mathematical viewpoint. It was not long before I decided to elaborate on my idea, first in my lecture on the Lorentz group from 1910… (Klein, *Gesammelte mathematische Abhandlungen*, 1921, p.413)

Klein’s work on projective geometry and group theory III

- In the course of Klein’s work metrical projective geometry provided:

1. A useful method to clarify the relations between geometry, number theory, and the theory of functions
2. A foundation of metrical geometry via Klein’s model of non-Euclidean geometry
3. A conceptual or, as Klein put it, “rational” foundation of special relativity (Klein, 1910, p.21).

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Klein’s sources

- Synthetic approach to projective geometry: Jakob Steiner, Christian von Staudt
- Analytic approach: August Ferdinand Möbius, Julius Plücker, Alfred Clebsch
- The theory of algebraic invariants: George Boole, James Joseph Sylvester, Arthur Cayley
- The transformation of the concept of space from a necessary presupposition to an object of research: Gauss, Riemann, Dedekind

A gap in von Staudt's treatment of projective geometry: Dedekind's axiom of continuity

Analytic geometers did not pay much attention to von Staudt's researches. This may have been because of the widespread idea that the essential aspect of von Staudt's geometry lies not so much in the projective approach as in the synthetic form. Von Staudt's considerations have a gap, which can only be filled by using an axiom, as described later on in the text. The same gap affects the extension of the method of von Staudt, as intended here. But our considerations concern not so much the extension as the original domain. The problem can be solved by specifying the analytical content of von Staudt's considerations, regardless of purely spatial ideas. Such a content can be summarized by demanding that projective space be represented by a numerical three-fold extended manifold. Besides, this is an assumption which lies at the foundation of any speculation about space. (Klein 1873, p.132, note)

Dedekind, *Stetigkeit und irrationale Zahlen* (1872)

The assumption of this property of the line is nothing else than an axiom by which we attribute to the line its continuity, by which we find continuity in the line. If space has at all a real existence it is not necessary for it to be continuous; many of its properties would remain the same even were it discontinuous. And if we knew for certain that space was discontinuous there would be nothing to prevent us, in case we so desired, from filling up its gaps, in thought, and thus making it continuous this filling up would consist in a creation of new point-individuals and would have to be effected in accordance with the above principle. (Dedekind 1901, p.12)

Klein’s “geometrical” interpretation of distance and the classification of geometries I

In order to determine the distance between two given points, I imagine them to be connected by a straight line. This line intersects the fundamental surface in two other points so that a cross-ratio of four points is constructed. I call the logarithm of this cross-ratio multiplied by a constant *c* the distance between the given points. (Klein 1871, p. 574)

- Euclidean and non-Euclidean metrics
  - Elliptic: Imaginary second-order surface.
  - Hyperbolic: The inner points of a real non-degenerate surface of second order.
  - Parabolic: The circle at infinity; one point is taken twice.

Klein’s definition of distance and the classification of geometries II

- Hesse’s principle of transfer: Suppose a manifold *A* has been investigated with reference to a group *B*, and by any transformation *A* is converted into A', then *B* becomes *B’* and the B'-based treatment of A' can be derived from the *B*-based treatment of *A* (Klein 1893, p. 72).
- Klein uses this principle to prove the equivalence of elliptic, hyperbolic, and parabolic geometries with the three cases of manifolds of constant curvature (i.e., $>$, $<$, and $= 0$, respectively) according to Beltrami’s “Teoria fondamentale degli spazi a curvatura costante” (1869).

From metrical projective geometry to the group-theoretical view

- *As long as our geometrical investigations are based on one and the same transformation group, the geometric content remains unvaried* (Klein 1893, p.73).
- Given a manifold and a group of transformations of it; to develop the theory of invariants relating to that group (in the 1921 edition of the Erlangen Program Klein refers here to his papers on non-Euclidean geometry)
- Geometrical properties are redefined as relative invariants of transformation groups.

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Klein, “Über die Arithmetisierung der Mathematik” (1895)

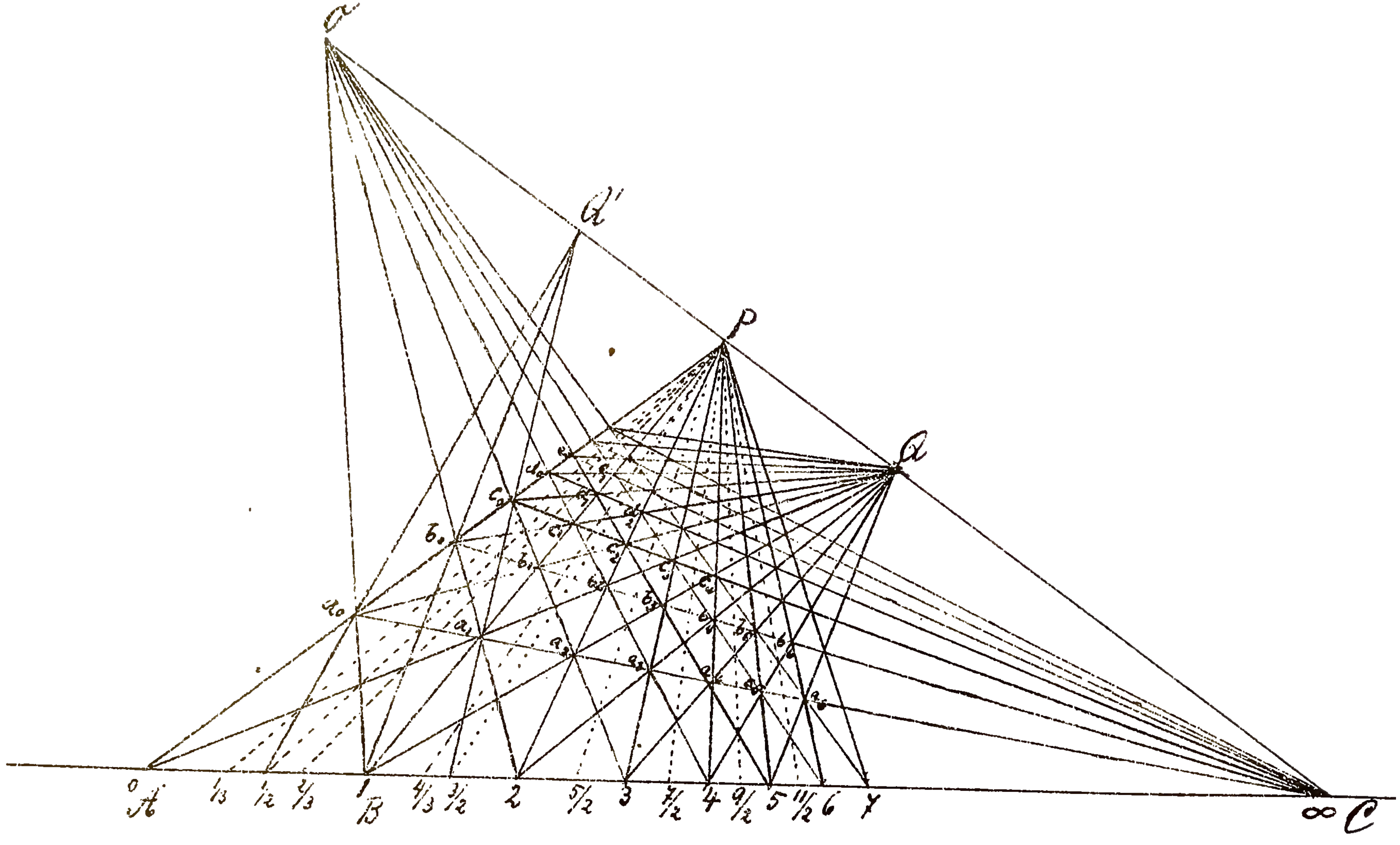
- Three roads to the arithmetization of mathematics
  - Weierstraß, Kronecker and the demand of a purely arithmetical way of proof
  - Peano school: A formal language for the different types of logical connections apart from association of ideas and vagueness
  - Gauß, Dedekind, Klein?

  The essential aspect for me does not lie in the arithmetical form of the development of thought, but in the logical rigor attained by this form. I consider this the positive side of my program of a new elaboration of other mathematical disciplines in analogy with the arithmetical foundation of the calculus. On the negative side, however, I have to point out that mathematics goes beyond the logical deduction. Besides the latter, intuition still retains its full and specific meaning. (Klein 1922, p.234)
  - Geometry, applications, and mathematics teaching

Klein's lectures on non-Euclidean geometry (1889-90) I

- Klein on Paul du Bois-Reymond's distinction between empiricism and idealism in the foundation of the calculus:
  - The representation of such abstract concepts as limits can be obtained only indirectly by the use of geometrical constructions
  - Limits exist as logical presuppositions of the calculus, although neither infinite nor infinitesimal quantities are imaginable in the sense of intuition
- Two approaches to the fundamental theorem of projective geometry:
  - Friedrich Schur: The concept of continuity is given immediately in intuition, which justifies the use of geometrical proofs.
  - Klein-Lüroth-Zeuthen and Darboux: The intuition is only the trigger for the logical formation of concepts, which has its model in the arithmetical definition of irrational numbers.

The construction of a numerical scale in Klein's lectures on non-Euclidean geometry



Klein's lectures on non-Euclidean geometry (1889-90) II

- Klein's distinction between pure and applied mathematics or mathematics of precision (calculation with real numbers) and mathematics of approximation (calculation with approximate values):
- Logical rigor is required in the foundation of mathematical theories; that which is given in intuition and experiment is approximate and subject to revision.

It is not so much a matter of inferring correct conclusions from correct premises, as to obtain those conclusions which follow with foreseeable correctness from approximately correct premises, or also to say to what extent further inferences can be made. (p. 314)

That which is mathematically sound, soon or later will have far-reaching implications beyond his original field. (Klein 1910, p.22)

Klein, “Zur Nicht-Euklidischen Geometrie” (1890) and the Clifford-Klein problem of space form

I consider an axiom to be the postulate, by which we read exact assertions into inexact intuition… Regarding the origin of axioms, I cannot say more than this: the abstraction that leads to them here as in other domains takes place unconsciously. Thereby, I can clarify my stance towards the theory of irrationals. Certainly, the construction of irrational numbers was triggered by the seeming continuity of spatial intuition. Since I do not attribute any precision to spatial intuition, however, I will not want the existence of irrationals to be derived from such an intuition. I think that the theory of irrationals should be developed and delimited arithmetically, to be then transferred to geometry by means of axioms, and hereby enable the precision that is required for the mathematical consideration. (p.572)

Klein’s lectures on non-Euclidean geometry (1928) III

Let us assume that the space about us exhibits a Euclidean or a hyperbolic structure. We can by no means infer from this that space has an infinite extent. Because, for instance, Euclidean geometry is entirely compatible with the hypothesis that space is finite, a fact that has been formerly overlooked. The possibility of ascribing a finite content to space whatever its geometrical structure, is particularly welcome because the idea of an infinite expanse, which was originally looked upon as a substantial progress of the human mind, gives rise to many difficulties, e.g. in connection with the problem of mass distribution (Klein [1889-90]1928, p.270; Engl. trans. in Torretti 1978, p.152).

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Russell, *An Essay on the Foundations of Geometry* (1897)

Since these systems are all obtained from a Euclidean plane, by a mere alteration in the definition of distance, Cayley and Klein tend to regard the whole question as one, not of the nature of space, but of the definition of distance. Since this definition, on their view, is perfectly arbitrary, the philosophical problem vanishes – Euclidean space is left in undisputed possession, and the only problem remaining is one of convention and mathematical convenience. (p.30)

- Distance, as a relation between two points, is part of our basic knowledge about space (p.36).
- "Projective coordinates are a set of numbers, arbitrarily but systematically assigned to different points, like the number of houses in a street, and serving only as convenient designations for points which the investigation wishes to distinguish" (p.119).
- Projective geometry as the a priori science of space

Cassirer vs. Russell (and Paul Natorp) in *Substanzbegriff und Funktionsbegriff* (1910) I

The geometrical axioms are not copies of the real relations of sense perception, but they are postulates by which we read exact assertions into inexact intuition. (p. 103)

The order of points of space is conceived in the same manner as that of numbers. It is true that the two fields remain strictly separated in essence; the "essence" of the figure cannot be immediately reduced to that of number. But precisely in this relative independence of the elements, as in the independence of their fundamental relation, is manifest the connection in general deductive method. As in the case of number we start from an original unit from which, by a certain generating relation, the totality of the members proceeds in fixed order, so here we first postulate a plurality of points and a certain relation of position between them, and in this beginning a principle is discovered from the various applications of which issue the to totality of possible spatial constructions….

Cassirer vs. Russell (and Paul Natorp) in *Substanzbegriff und Funktionsbegriff* (1910) II

In this connection, projective geometry has with justice been said to be the universal "a priori" science of space, which is to be placed besides arithmetic in deductive rigor and purity. Space is here deduced merely in its most general form as the "possibility of coexistence" in general […] by the addition of special completing conditions, the general projective determination can be successively related to the different theories of parallels and thus carried into the special "parabolic," "elliptic" or "hyperbolic" determinations. (p.88)

- Cassirer on the group-theoretical view: The only implicit assumption that lies at the foundation of geometry is "a system of possible transformations" rather than immutable geometrical properties.

Cassirer’s later remarks on arithmetization I

Here again it is unnecessary to introduce the unreal elements as individuals leading some sort of mysterious existence side by side with the real points; the only logically and mathematically meaningful statement that can be made about them refers to the permanence of the relations that are embodied and expressed in them. But of course the symbolic thinking of mathematics does not content itself simply with apprehending these relations in abstracto; it demands and creates a special sign for the logical and mathematical relationship that is present in them and ultimately treats the sign itself as a fully valid, legitimate, mathematical object. (Cassirer 1929/1957, *The Philosophy of Symbolic Forms*, vol. 3, p.397)

Cassirer's later remarks on arithmetization II

In his historical survey of the development of mathematical thought during the nineteenth century Felix Klein declared that one of the most characteristic features of this development is the progressive "arithmetization" of mathematics. Also in the history of modern physics we can follow this process of arithmetization. From Hamilton's quaternions up to the different systems of quantum mechanics we find more and more complex systems of algebraic symbolism. The scientist acts upon the principle that even in the most complicated cases he will eventually succeed in finding an adequate symbolism which will allow him to describe his observations in a universal and generally understandable language. (Cassirer, *An Essay on Man*, 1944, p.276)

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Concluding Remarks I

- Klein and Dedekind’s logical structuralism (cf. Reck 2003)
- Set-theoretic structuralism
  - Sets as primitive
  - Functions?
- Ante rem structuralism
  - Platonism
  - Extensibility?
- Structuralism in category theory
  - A systematic classification of geometries
  - The “formal supervenience” of group-theoretical properties upon geometric properties

Concluding Remarks II

Klein was well aware of the alternative: it is possible to put the transformation group first and define a geometry afterwards *or* start from a geometry given in one way or another and then consider its transformation group. In the latter case, the relationship between geometric properties and transformation groups can be put in terms of a relationship which has attracted the attention of philosophers over the last twenty-five years, although in fields which have nothing to do with mathematics and logic. Indeed, I claim that one can say that group-theoretical properties *formally supervene* upon geometric properties. (Marquis 2009, p.36)

- A non-reductionist view of arithmetization
  - Cf. “Abstraction” (and “creation”) in neo-Kantianism
  - Mathematical practice and heuristics

Thank you for your attention

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