Questions of Generality as Probes into Nineteenth-Century Mathematical Analysis

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> At the beginning of the century, the idea of a function was a notion both too narrow and too vague. … It has all changed today; one distinguishes between two domains, one is limitless, the other one is narrower but better-cultivated. The first one is that of functions in general, the second one that of analytic functions. In the first one, all whimsies are allowed and, every step of the way, our habits are clashed with, our associations of ideas are disrupted; thus, we learn to distrust some loose reasoning which seemed convincing to our fathers. In the second domain, those conclusions are allowable, but we know why; once a good definition had been placed at the start, rigorous logic reappeared. [1]

In this passage from Poincaré's 1898 eulogy of Weierstrass, the French mathematician gave his version of the classical description of the rise of rigor in mathematical Analysis[2] in the nineteenth century. Though the quotation is quite straightforward, two elements raise questions. First, it is customary to associate vague ideas with limitless object-domains, and precise definitions with clearly bounded object-domains[3]; conceptual clarification walks hand in hand with domain-restriction. However, Poincaré described here the passage from a vague to a distinct notion of function as walking hand in hand with domain-extension. Second, after reading the last sentence, one would expect Poincaré to give this "good definition" which took one century to emerge and whose emergence eventually put mathematical Analysis back on the safer track of rigor. However, this definition is nowhere to be found in Poincaré's paper and, as we will see, this feature is in no way specific to Poincaré: the "function in general" is not something one defines but something one points to; it is not an object to be studied but the background on which objects can be studied. These topics will be discussed in the first part of this chapter.

This first part will provide the background against which we will endeavor to delineate two other historical interactions between generality issues and function theory in the nineteenth century. We will first focus on the first years of the nineteenth century and use questions of generality to attempt a comparison between two major treatises on function theory, that of Lagrange and that of Cauchy. We will attempt to show how Lagrange and Cauchy chose different strategies to take up the same three-fold generality challenge: to give a general (uniform) account of the general behavior (that is, save for isolated values of *x*) of a general

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[1] Au commencement du siècle, l'idée de fonction était une notion à la fois trop restreinte et trop vague. … Aujourd'hui tout est bien changé; on distingue deux domaines, l'un sans limites, l'autre plus restreint, mais mieux cultivé. Le premier est celui de la fonction en général, le second celui de la fonction analytique. Dans le premier, toutes les fantaisies sont permises et à chaque instant nos habitudes sont heurtées et nos associations d'idées rompues ; nous y apprenons ainsi à nous défier de certains raisonnements par à peu près qui paraissaient convaincants à nos pères. Dans le second, au contraire, ces conclusions sont permises ; mais nous savons pourquoi ; il a suffi de placer au début une bonne définition ; et on a vu reparaître une rigoureuse logique. (Poincaré 1899: 4)

[2] For the sake of clarity, we will systematically write « Analysis » with a capital A to denote mathematical Analysis (function theory). Thus, we will present an analysis of Analysis.

[3] To prevent any misunderstandings: the objects referred to here are functions, the object-domains are function classes or function-sets.

(non-specified) function. On the basis of the elements gathered in the first two parts of this chapter, we will sketch a systematic comparison grid.
In the third part, we will concentrate on the end of the nineteenth century so as to show how some mathematicians used the sophisticated point-set theoretic tools provided for by the advocates of rigor to show that, in some way, Lagrange and Cauchy had been right all along: counter-revolution, as we all know, is a synthesis of pre-revolutionary and revolutionary elements. On the basis of the mathematical material covered in this third part, we will put forward a new concept, that of *embedded* generality. Though we came across it in the context of analysis, it is by no means specific to that context. We will argue that it captures an approach to generality issues that is specific to mathematics and whose mathematical treatment is a striking feature of twentieth-century mathematics.
Before we start, a few remarks on the nature of this chapter. To someone who usually works as a historian of mathematics, it may appear somewhat quick-paced: the various contexts are barely sketched; the collection of quotations displays a kind of imaginary dialogue between mathematicians, regardless of actual historical connections. However, it must be acknowledged that the main goal of this chapter is of an epistemological nature[4], and that the audience intended is not primarily that of historians. We aim at documenting ways of *expressing* generality and *epistemic configurations* in which generality issues became linked with other topics, be they epistemological topics such as rigor or mathematical topics such as point-set theory. In this regard, we present and try to characterize three very specific configurations: the first one evolving from Abel to Weierstrass, the second one in Lagrange's Treatises on analytic functions, the third one in Borel.

1. Generality, rigor, arbitrariness.

1.1 Abel's letter to Hansteen.

We can start by reading an excerpt from one of Abel's letter to his master Hansteen, written in 1826:

> I shall devote all my strengths to shedding some light on the immense obscurity which, at present, reigns over analysis. It is so devoid of plan and system that one is astonished by the fact that so many people indulge in it – and, what is even worse, it lacks rigor, absolutely so. In higher Analysis very few propositions are proved with conclusive rigor. Everywhere, we come across the sorry habit of concluding from the special to the general and, what is amazing is that, after such a procedure, one rarely find what is called a paradox. The reason for that is indeed very interesting to think over. The reason, to my mind, lies in the fact that most of the functions dealt with by Analysis up to now can be expressed by powers. When other ones mingle with them, which, admittedly, seldom occurs, we don't do so well; were one to draw false conclusions, from them would spring an infinity of tainted propositions, all standing together. [5]

[4] Here we use the adjective « epistemological » with the meaning it has in the French tradition, denoting what pertains to the theory of science and not what pertains to the theory of knowledge in general.
[5] Je consacrerai toute mes forces à répandre de la lumière sur l'immense obscurité qui règne aujourd'hui dans l'analyse. Elle est tellement dépourvue de tout plan et de tout système, qu'on s'étonne seulement qu'il y ait tant de gens qui s'y livrent – et, ce qui pis est, elle manque absolument de rigueur. Dans l'Analyse supérieure bien peu de propositions sont démontrées avec une rigueur définitive. Partout on trouve la malheureuse manière de conclure du spécial

This quotation nicely parallels that of Poincaré which we gave in the introduction: Poincaré looked backward on a century-long process which Abel, among others, had kicked off. The themes of rigor and generality are here beautifully intertwined, in a way which, to a large extent, will prove stable throughout the nineteenth century. We need to distinguish between two levels, the epistemic level and the object-level. On the epistemic level, this quotation is famous for its ideal image of mathematics (or its image of ideal mathematics) as a set of interrelated theorems: if one false assertion is mistakenly taken to be true, then the whole network is tainted. As a consequence, the current lack of rigor in "higher Analysis" – that is infinitesimal and integral calculus – is an outrage. But the situation described by Abel is a paradoxical one. One the one hand, logic tells us that there *might* be false assertions in analysis, since the usual mode of reasoning is in itself faulty: "concluding from the special to the general"; mathematical truth cannot rely on induction. On the other hand, it appears that there *are not* so many false assertions or pairs of contradictory assertions as one could expect: the body of knowledge, however ill-founded, seems to be sound and safe. The reason for this is sought for on the object-level: the objects mathematicians usually consider – Abel writes – are functions of a special kind, namely functions which can be expressed as power series (with positive and negative integer powers and, on occasion, fractional powers as well). On this basis, two different lines of research can emerge, both of which require specific proof-methods. One can either stick to the study of this special class of functions, or try to understand how more general functions behave. To study the links between generality and rigor we need to take a closer look at the second line of research, in which, as we shall see, generality in the object-domain is expressed thanks to a notion of *arbitrariness*.

1.2 Investigating the generality of a theorem: Dirichlet 1829.

Our next landmark in the history of rigor in function theory is Dirichlet's 1829 article *Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données*. Let us quote the final paragraphs of this fifteen-page article. The flag of rigor is first waved, then the main result summarized:

> The former considerations prove in a rigorous way that, if the function $\varphi(x)$, all values of which are assumed to be finished and determined, shows but a finite number of discontinuities between the limits $-\pi$ and π, and, moreover, has but a definite number of maxima and minima between these limits; series (7)[6], the coefficients of which are definite integrals depending on function $\varphi(x)$, is convergent and takes on a value whose general[7] expression is
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> $$\tfrac{1}{2}\,[\varphi(x+\varepsilon) + \varphi(x-\varepsilon)],$$
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> where ε stands for an infinitely small number.[8]

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au général, et ce qu'il y a de merveilleux, c'est qu'après un tel procédé on ne trouve que rarement ce qu'on appelle des paradoxes. Il est vraiment très intéressant de rechercher la raison de ceci. Cette raison, à mon avis, il faut la voir dans ce que les fonctions dont s'est jusqu'ici occupée l'analyse, peuvent s'exprimer pour la plupart par des puissances. Quand il s'y en même d'autres, ce qui, il est vrai, n'arrive pas souvent, on ne réussit plus guère, et pour peu qu'on en tire de fausses conclusions, il en naît une infinité de propositions vicieuses qui se tiennent les unes les autres. (Abel 1892: 263)

[6] The Fourier series.

[7] In this context, « general » means for every value of *x*.

[8] Les considérations précédentes prouvent d'une manière rigoureuse que, si la fonction $\varphi(x)$, dont toutes les valeurs sont supposées finies et déterminées, ne présente qu'un nombre fini de solutions de continuité entre les limites $-\pi$ et π, et si en outre elle n'a qu'un nombre déterminé

This is what students still learn today as “Dirichlet’s theorem”: a 2π-periodic function which is piecewise continuous and piecewise monotonous has a converging Fourier series (series (7) in the quotation), whose limit is $\varphi(x)$ if φ is continuous at x, and, more generally, the mean value of $\varphi(x^-)$ and $\varphi(x^+)$. Obviously, the theorem says nothing about the “arbitrary function”, quite the contrary: the hypotheses under which the conclusion has been proved to hold are painstakingly spelled out, which is exactly what being rigorous means. This theorem took ten pages to prove, and each of the hypotheses played a part in at least one step of the proof. One interpretation could be that Dirichlet managed to prove that, contrary to what Fourier had assumed, arbitrary (periodic) functions – that is any periodic function, whether encountered in a purely mathematical context or in a context of mathematical physics – cannot be represented by a trigonometric series … end of the story. The end of Dirichlet’s paper shows that it is not quite so:

> We still have to investigate the cases in which what we have assumed as to the number of discontinuities and that of maxima and minima ceases to be the case. [9]

Dirichlet managed to prove the conclusion under some hypotheses which emerged in the proof, but acknowledged the fact that this conclusion may hold under weaker hypotheses. Another way of saying this (but we should notice that Dirichlet never spoke of function classes or function sets) is: this theorem asserts that a given property is valid for some object-domain, but it is likely that this object-domain can be extended; it is likely – or at least worth investigating – that the conclusion is still valid for more general functions. The end of Dirichlet’s paper pointed to two ways of exploring the generality of the conclusion, that is, the extent of the domain of objects for which it holds. Dirichlet first wrote that the proof could be amended for functions with an infinite number of discontinuities as long as the set of points of discontinuities is nowhere dense (in modern parlance)[10]. The latter restriction came from the fact that the coefficients of the Fourier series are integrals involving a function φ, and that the very notion of an integral becomes meaningless if restrictions are not put on the set of points of discontinuity. We can see that, in order to asses the generality of a conclusion which he had established under what he felt to be too strong hypotheses, Dirichlet first resorted to *proof analysis*[11], but this proof analysis led him to the analysis of a mathematical concept, that of integrable function. In this 1829 paper, he merely pointed to this concept analysis as a research program: “But, doing things with as much clearness as one can wish for demands that one go into some details as to the fundamental principle of infinitesimal analysis; these details will be expounded in a further note …” [12]. A few lines above, Dirichlet used a different strategy to explore the generality of the property. Instead of pointing to the general concept of the integral and a possible weakening of the hypotheses under which integrability can be

de maxima et de minima entre ces mêmes limites, la série (7), dont les coefficients sont des intégrales définies dépendant de la fonction $\varphi(x)$, est convergente et a une valeur généralement exprimée par : ½ $[\varphi(x+\varepsilon) + \varphi(x-\varepsilon)]$, où ε désigne un nombre infiniment petit. (Dirichlet 1829: 168)

[9] Il nous reste à considérer les cas où les suppositions que nous avons faites sur les nombre de solutions de continuités et sur celui des valeurs maxima et minima cessent d’avoir lieu. (Dirichlet 1829: 168)

[10] We need not discuss here the relevance of this integrability condition.

[11] Our analysis of Dirichlet’s move is, of course, very close to Lakatos’ (Lakatos 1976: 148-149).

[12] Mais la chose, pour être faite avec toute la clarté qu’on peut désirer, exige quelques détails liés aux principes fondamentaux de l’analyse infinitésimale et qui seront exposés dans une autre note … . (Dirichlet 1829: 169)

ascertained, he gave an example of a function that is *too* arbitrary to belong to the maximal object-domain for which the conclusion holds, more precisely, too arbitrary to be integrable:

> One would get an example of the function which doesn't fulfil this requirement if one assumed that $\varphi(x)$ equalled some determined constant c when variable x takes on a rational value, and equalled some other constant d when the variable is irrational. [13]

This strange function had not until then been considered in mathematical Analysis; it had not turned up so far, whether in pure mathematics or in mathematical physics. This *display* of a specific function is an element of a new *mathematical configuration* in function theory, a configuration which also encompasses epistemic values such as "rigor", epistemic practices [14] such as proof-analysis, but also strictly mathematical elements such as the exploration of the various properties of points sets on the real straight line. The "Dirichlet moment" in the theory of Fourier series is perfectly characterized by Riemann in his 1854 dissertation on the same topic:

> The works on this question which we have so far mentioned endeavored to establish the Fourier series for those functions which are encountered in mathematical physics; thus, one could start the proof for completely arbitrary functions and, then, put any restriction as to the course of the function as were necessary for the proof, so long as these restrictions don't go against the purpose. [15]

We need not remind the reader that Riemann was Dirichlet's student and that the goal of this 1854 dissertation was to fulfil the research program which Dirichlet had sketched at the end of his 1829 paper.

1.3 Expressing generality through arbitrariness

Before studying further the specific links between generality, rigor and arbitrariness in this new mathematical practice, we need to pay more attention to the way general/arbitrary functional objects were referred to by mathematicians. We shall distinguish between three modes of expression.

1.3.1 Referring to.

From its emergence as an autonomous mathematical concept in the eighteenth century, an element of arbitrariness had always been part of the function concept, however quickly the concept may have been sketched. As for functions which turned up in purely mathematical contexts, they could be either formed by the free juxtaposition of symbols (a freedom subject to syntactic constraints, however) or given (in the case of continuous functions) by a freely drawn plane curve. Both ideas were to be found in Euler's 1748 *Introductio in Analysin Infinitorum*, in the first and second part respectively, but they were nothing more than a way of introducing the reader to the kind of mathematical situations that they will be taught to

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[13] On aurait un exemple d'une fonction qui ne remplit pas cette condition, si l'on supposait $\varphi(x)$ égale à une constante déterminée c lorsque la variable x obtient une valeur rationnelle, et égale à une autre constante d, lorsque cette variable est irrationnelle. (Dirichlet 1829: 169)

[14] We will use "practice" instead of "configuration" when we assume a more agent-based approach.

[15] Die bisherigen Arbeiten über diesen Gegenstand hatten den Zweck, die Fourier'sche Reihe für die in der Natur vorkommenden Fälle zu beweisen; es konnte daher der Beweis für eine ganz willkürlich angenommene Function begonnen, und später der Gang der Function behuf des Beweises willkürlichen Beschränkungen unterworfen werden, wenn sie nur jenen Zweck nicht beeinträchtigen. (Riemann 1892: 244)

handle. Questions of generality and arbitrariness were neither central nor even marginal in this context. In mathematical physics however, the theory of vibrating strings and the subsequent development of Fourier theory placed the question of the mathematical description of the "arbitrary function" (*fonction arbitraire*) – describing a physical phenomenon – on the center stage, as we saw earlier.

In terms of vocabulary, throughout the nineteenth century, "arbitrary function" remained very much in use: in Fourier's 1822 *Théorie analytique de la chaleur*, in Dirichlet's 1829 paper. In his 1837 paper, Dirichlet used the German translation "willkürlich" (arbitrary) and, as if "arbitrary" would not convey the idea with sufficient strength, sometimes used "ganz willkürlich" (completely arbitrary); he also wrote "ganz gesetzlos" (completely lawless). The same words were to be found in Riemann's 1854 paper: "willkürlich", "ganz willkürlich", sometimes "ohne besondere Voraussetzungen über die Natur der Function"[16] (without any specific hypothesis as to the nature of the function). In his 1875 paper on the classification of functions, Paul du Bois-Reymond turned it into a adjective: "die voraussetzungslose Function" (the hypotheses-free function). The very same year, in France, "arbitraires" was replaced by "les plus générales" (most general) in Darboux's *Mémoire sur les fonctions discontinues*. The opening paragraph reads:

> At the risk of being too long, I was set on being rigorous, perhaps without full success. Many points which would justly be considered obvious or would be granted in the applications of science to usual functions [*fonctions usuelles*], have to undergo rigorous criticism when it comes to expounding the propositions pertaining to the most general functions.[17]

1.3.2 Describing

The easiest way to describe a general/arbitrary function is to negate a property which you feel to be specific. For instance, in his 1837 paper, Dirichlet shortly explained what his "gesetzlos" meant: "it is by no means necessary to assume that the dependence [between the function and the variable] is expressible with mathematical operations" [18]; here, Dirichlet negated one of the two standard descriptions of what a function is. In the same paper, Dirichlet pointed to arbitrariness by negating another property which is usually (more or less implicitly) assumed: *usual* functions (in modern parlance: analytic functions) are completely determined by their behavior in any interval belonging to their domain of analyticity (uniqueness of analytic continuation); thus for an arbitrary function, "so long as one has determined the function for only a part of the interval, its continuation for the rest of the interval remains completely arbitrary."[19]

Alongside the negation of a property encountered in *usual* functions, more positive descriptions can be found. Here, a *generic mode of description* was used. The standard one, at

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[16] The 20th century German spelling would be « *Funktion* ».

[17] Au risque d'être trop long, j'ai tenu avant tout, sans y réussir peut-être, à être rigoureux. Bien des points, qu'on regarderait à bon droit comme évidents ou que l'on accorderait dans les applications de la science aux fonctions usuelles, doivent être soumis à une critique rigoureuse dans l'exposé des propositions relatives aux fonctions les plus générales. (Darboux 1875: 58)

[18] man braucht nicht einmal an eine durch mathematische Operationen ausdrückbar Abhängigkeit zu denken (Dirichlet 1889 135)

[19] so lange man über eine Function nur für einen Theil des Intervalls bestimmt hat, bleibt die Art ihrer Fortsetzung für das übrige Intervall ganz der Willkür überlassen. (Dirichlet 1889: 136)

least for continuous function, remained that of the arbitrarily drawn curve: in spite of the radical change in epistemic configurations, the description remained stable from Euler to Dirichlet and Riemann (e.g. "the arbitrary (graphically given) functions"[20] in Riemann). A significant change occurred in 1875 in du Bois-Reymond's paper on the classification of arbitrary functions; to introduce the most general function concept, that of the function "on which no hypotheses are made", he discarded the classical "arbitrary curve" image (suited for continuous functions only):

> I. The hypotheses-free function.
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> In the case where no specific determination presents itself, the mathematical function is a table – similar to an ideal logarithmic table, thanks to which to any specified numerical value of the independent variable, one or several functional values – or one indeterminate but between limits given in the table – is associated. No horizontal row of the table has any influence on any other, that is, each and every value in the column which displays the values of the function stands by itself and may be altered; such an alteration would not prevent the column from representing a mathematical function. The mathematical concept of functions holds nothing more (nothing less either); it is thus fully exhausted. [21]

By a strange turn of events, one of the standard modes of description for the most classical and specific functions – one which pre-dates by centuries the emergence of the function concept – is used to express how a general function can be given: when dealing with usual functions such as logarithms or sines, tables of values are commonly used; a general function can be given by a similar table, but an "ideal table", in which every value is completely independent from the others (which is, of course, reminiscent of the negation of the uniqueness of continuation). With this generic description, du Bois-Reymond was close the twentieth-century concept of a map between sets, though he didn't require that two sets be declared beforehand.

1.3.3 Exemplifying.

Exemplifying the general: the endeavor sounds paradoxical. Following Nelson Goodman, we shall say that to be displayed as a *sample*, an "object" has to both possess and denote (or refer to) a property [22]. Thus, a *sample* of generality is a contradiction in terms, since no individual object – be it of a *usual* or of an *extraordinary* kind – can posses the property of being general. However, in the epistemic configuration which links generality, rigor and arbitrariness in nineteenth-century function theory, mathematicians pointed to generality by displaying examples; we saw one of the first instances in Dirichlet's 1829 paper, with the example of (to rephrase) the indicatrix of the set of rational numbers within the set of real

[20] die willkürlichen (graphisch gegebenen) Function. (Riemann 1892: 227)

[21] I. Die Voraussetzunglslose Function. Die mathematische Function, falls keine besondere Bestimmungen für sie vorliegt, ist eine den Logarithmentafeln ähnliche ideale Tabelle, vermöge deren jedem vorausgesetzten Zahlenwerthe der unabhängige Veränderlichen ein Werth oder mehrere, oder ein zwischen Grenzen, die in der Tabelle gegeben sind, unbestimmter Werth der Function zugehört. Keine Horizonalereihe der Tabelle had irgend einen Einfluss auf die anderen, d.i. jeder Werth in der Columne der Functionalwerthe besteht für sich und kann für sich geändert werden, ohne dass die Columne aufhört eine mathematische Function darzustellen.
Mehr enthält der Begriff der mathematischen Function nicht und auch nicht weniger, er ist damit vollig erschöpft. (du Bois-Reymond 1875: 21)

[22] (Goodman 1976: 53)

numbers. The exemplification tactics differed in a striking way from the two we have described so far. When it came to referring or describing what a function of the most general type could be, mathematicians strove for the least specific (whether by negating common but specific properties or by describing generic *templates* to be filled arbitrarily). When examples are to be displayed in order to point to the general, the more specific the example, the more successful the denotation. We need not go into the details of the history of "pathological", "bizarre" (Borel), "amusing" ("*drôlatiques*" in Darboux[23]) functions; let us just mention Riemann's example of a (Riemann-)integrable function whose set of discontinuities is dense (1854), Weierstrass' continuous but nowhere differentiable function (1872) and Hankel's monster-function producing process (based on his "principle of condensation of singularities") [24].

1.4 Arbitrary functions: what for ?

Examples of functions with extraordinary properties are sometimes used as counter-examples, but their display can fulfil other purposes. The mere displaying of the "monster" reveals a new and unexpected feature of the function-world. Geometric intuition is the first victim of this display:

> Indeed, the existence of the derivative in a continuous function $f(x)$ is reflected geometrically by the existence of a tangent line at any point of the continuous curve which is the geometric image of this function; and, though it is possible for us to conceive that at some singular points, even very close one to the other, the direction of the tangent line be parallel to the x-axis or to the y-axis, or even completely indeterminate, we cannot conceive that it be so in every arc of the curve, however small it may be taken. Hence the tendency to consider it unnecessary to prove the existence of the derivative in a continuous function. [25]

This quotation, by Belgian mathematician Gilbert, is also here for the sake of irony: Gilbert is (somewhat) famous for his attempt to prove that a continuous function is piecewise differentiable, the very same year Weierstrass displayed (in the Berlin Academy of Science) his nowhere differentiable continuous function ! In this passage, his goal was to remind the reader that intuition is no adequate ground for mathematical knowledge and that, consequently, this differentiability property called for a proof in spite of its intuitive nature; he certainly didn't mean to underline the deceiving nature of geometric intuition when general (continuous) functions were being considered. It has to be noted that, contrary to the first examples such as the indicatrix of **Q**, Weierstrass' function is highly sophisticated: you don't just come across it, and once the formula is written down it still takes skilful mathematical work to establish the nowhere differentiability … like rare and exotic butterflies, monster-functions can be hard to track down. The know-how in the monster-making business is definitely a part of the mathematical practice which we're endeavoring to delineate.

More generally, the display of specific functions with unusual properties is a tool for the assessment of the generality of a given statement. For instance, the example of the indicatrix of **Q** showed that the Fourier-series development process is not universally valid: the theorem proved in 1829 established its validity for a given (presumably not maximal) class of functions, and the example showed that the maximal class couldn't be all-encompassing. The display of the monster helped point to the task of identifying the exact contours of the right

[23] Quoted in (Gispert 1983: 83)

[24] For an analysis of nineteenth-century teratology (i.e. science of monsters) in function theory, see for instance (Volkert 1987)

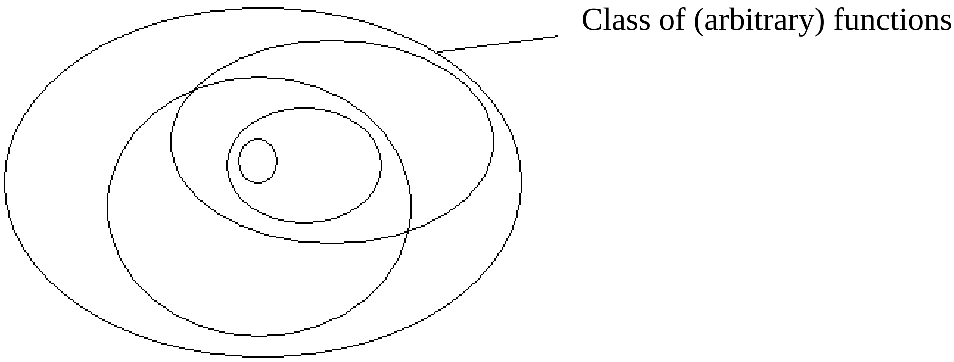
[25] Quoted in (Volkert 1988: 218). My translation.

function-class. On a more general level, we endorse Klaus Volkert's interpretation of the monster-displaying business [26]: pathological functions served as milestones for the *extensional* exploring of the function-world. We can read an explicit description of this way of charting the function-world in du Bois-Reymond's paper:

> First come a number of conditions satisfied by a function over a whole interval – however small; each of the conditions in the series restricts the function ever more, so that every former function classes encompasses all the following ones – with functions always assumed to be finite. [27]

Fifty years after Abel's lament about the complete lack of "plan" and "system" in mathematical Analysis, a form of systematicity had emerged: functions are grouped in classes, function classes are characterized by explicit (set-theoretic) properties; logical implications between properties (on the *intentional* level) are reflected on the *extensional* level by inclusion relations between function classes. This systematic way of charting the world of functions is typical of du Bois-Reymond's work (arbitray functions $\supset$ integrable functions $\supset$ continuous functions $\supset$ differentiable functions) or of Camille Jordan's *Traité d'Analyse* [28], whose second edition is a landmark in the history of "rigorous" analysis.

This interpretation also helps us understand the role of the "arbitrary function". As du Bois-Reymond strikingly put it, the "most general" function, the "arbitrary function", the "function on which no hypotheses is made" is something about which nothing can be said[29]. It is by no means an object to be studied, it is but an (intentionally) empty place in the whole epistemic configuration: not something to investigate, but a kind of background against which ever more specific function classes can be delineated; meaningless (on the intentional level) because all-encompassing (on the extensional level). This interpretation also helps us clarify the relationship between the general/arbitrary function and the pathological examples. Both are necessary elements of the same epistemic configuration, which doesn't mean that pathological functions serve as samples for the class of arbitrary functions – a part which, as noted earlier, no example can play.



[26] (Volkert 1987)

[27] Es gibt zuvörderst eine Anzahl von Bedingungen, die für ein ganzes wenn auch beliebige kleines Intervall einer Function gelten, und von denen jede folgende die Function immer mehr einschränkt, so dass jede vorhergehende Klasse von Functionen alle folgenden enthält, wobei die Function durchweg endlich angenommen werden. [du Bois-Reymond 1875: 21]

[28] (Jordan 1991)

[29] As mentioned earlier, du Bois-Reymond's general function concept is close to the abstract map concept but differs slightly. Things can be studied in an abstract map: is it one-one, is it onto etc. du Bois-Reymond fails to see these questions since the sets between which the map works are still implicit in his approach to the general function concept.

A few years later, the same charting of the function-world was used by Hilbert in his famous 1900 Paris address. Before expounding the last series of problems (pb. 19-23), a series devoted to problems in mathematical analysis, he discussed the relevance of various function classes; let us just read the first few lines, a wonderful sample of this fine tuning of the relevant class within the new "system" of functions:

> If we look over the development of the theory of functions in the last century, we notice above all the fundamental importance of that class of functions which we now designate as analytic functions – a class of functions which will probably stand permanently in the center of mathematical interest.
>
> There are many different standpoints from which we might choose, out of the totality of all conceivable functions, extensive classes worthy of a particularly thorough investigation. Consider, for example, the class of functions characterized by ordinary or partial algebraic differential equations. It should be observed that this class does not contain the functions that arise in number theory and whose investigation is of the highest importance. …
>
> If, on the other hand, we are led by arithmetical or geometrical reasons to consider the class of all those functions which are continuous and indefinitely differentiable, we should be obliged in its investigation to dispense with that pliant instrument, the power series, and with the circumstance that the function is fully determined by the assignment of values in any region, however small. While, therefore, the former limitation of the field of functions was too narrow, the latter seems to me too wide. [30]

2. General theory as a theory of the general behavior of a function: Lagrange and Cauchy.

Going backward in time and focusing on texts which don't belong to the epistemic configuration that we studied earlier, we come across another lead: other links between questions of generality and the historical development of mathematical Analysis in the nineteenth century appear. We will thus follow two leads at a time: on the one hand, we will try to characterize the various ways in which Lagrange's grasp of the world of functions differs from the one we described in the former paragraph, thus delineating two (ideal) *mathematical configurations*; on the other hand, we will come across a new conceptual intersection between questions of generality and the theory of function: once a function is given, one can try to distinguish between a *general behavior* (to be studied in a uniform way) and points where the behavior is singular (to be investigated later, with specific tools). We will see that this idea of general behavior of a function is common to both Lagrange and Cauchy, but treated in very different ways by these mathematicians: we think this point of comparison is quite illuminating and helps understand some of the peculiarities of Cauchy's concept of continuity.

2.1 Scenes from Lagrange's Théorie des fonctions analytiques.[31]

The introduction to Lagrange's treatise is entitled "des fonctions en général" (on functions in general):

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[30] (Hilbert 1902: 467)

[31] As the title of this paragraph indicates, we certainly do not mean to give an overview of Lagrange's work and its relationship to questions of generality in mathematics or mathematical physics. For a detailed study of the conceptual architecture of Lagrange's *Théorie des fonctions analytiques*, see (Ferraro and Panza).

> One calls *function* of one or several quantities any calculating expression in which these quantities appear in any way, along with other quantities which are considered to have given fixed values, in contrast to the quantities in the function which may take on any possible values. … The word *function* was used by the first analysts to denote generally the powers of a given quantity. Since then, the meaning has been extended to any quantity formed in any way from another quantity. [32]

As mentioned earlier, an element of arbitrariness is present in Lagrange's function concept, but as a part of a very different configuration. The distinction between *axiomatic* and *genetic* definitions can help us contrast Lagrange against, say, du Bois-Reymond. Du Bois-Reymond needed a definition (or, at least, a template) for the most general/arbitrary function; this concept had maximal extension and minimal intention, but was necessary for the definition of more interesting function classes in terms of characteristic properties: the most general function – this nondescript element of the class of all functions – was the starting point for the systematic exposition of mathematical Analysis. The notion of function in Lagrange is a genetic one: the basic, simple elements are known (letters standing for variable quantities) and they are to be combined at will to form any function you like; of course, the free combination of symbols has to remain within certain syntactic bounds, but for those as well only the most simple ones are known (namely, the general rules of algebra and maybe the symbols for the derivative, the partial derivative and the integral): just as new functions can be formed, it is quite possible to add new syntactic structures. There is no need in Lagrange for a definite criterion enabling us to distinguish between functions and non-functions, no need to precisely delineate the outer rim of the function world. Quite the contrary, the function world is an open-field; generality, a mere horizon. The challenge is to find a systematic way to study these functional objects, of which the basic elements (in generic terms) but not the basic properties (in axiomatic terms) are known.

Lagrange met this challenge by resorting to a general mode of description, a general "form":

> Let us consider therefore a function $f(x)$ of any variable x. If x is replaced by $x+i$, i being any indeterminate quantity, the function becomes $f(x+i)$ and, thanks to the theory of series, it will be possible to develop it into a series of the following type
>
> $$f(x) + pi + qi^2 + ri^3 + \ldots,$$
>
> in which quantities p, q, r…, the coefficients of the powers of i, are new functions of x, derived from the primitive function of x and independent of indeterminate i. [33]

Whatever the form of $f(x)$, seen as an formula in which x appears, it can be written in the *universal form* of a power series. The generality strategy is clear, but the claim remained to be ascertained:

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[32] On appelle fonction d'une ou de plusieurs quantités toute expression de calcul dans laquelle ces quantités entrent d'une manière quelconque, mêlée avec d'autres quantités qu'on regarde comme ayant des valeurs données et invariables, tandis que les quantités de la fonction peuvent recevoir toutes les valeurs possibles. … Le mot fonction a été employé par les premiers analystes pour désigner en général les puissances d'une même quantité. Depuis, on a étendu la signification de ce mot à toute quantité formée d'une manière quelconque d'une autre quantité … (Lagrange 1813: 15)

[33] Considérons donc une fonction $f(x)$, d'une variable quelconque x. Si à la place de x on y met $x+i$, i étant une quantité quelconque indéterminée, elle deviendra $f(x+i)$, et, par la théorie des séries, on pourra la développer en une série de cette forme $f(x) + pi + qi^2 + ri^3 + \ldots$,dans laquelle les quantités p, q, r …, coefficients des puissances de i, seront de nouvelles fonctions de x, dérivées de la fonction primitive de x et indépendantes de l'indéterminée i. (Lagrange 1813: 21)

> But to avoid advancing anything gratuitously, we shall examine the very form of the series which is to represent the development of any function *f*(x) when $x+i$ is substituted for x and in which we have assumed only positive, whole powers of x appear. This requirement is indeed met be the development of the various known functions; but no one, to my knowledge, has ever tried to establish it *a priori*, what seems all the more necessary since there are particular cases in which it might fail to be met. [34]

According to Lagrange, a proof for the generality of the property is all the more needed since counter-examples were known … it could be the oddest justification for the need of a general proof ever given ! This quotations reveals a feature of the epistemic configuration to which Lagrange belongs, a feature that we haven't encountered so far:

> I will first prove that, in the series which results from the development of function *f*($x+i$), no fractional power of i can appear, unless x takes on some particular values. … This proof is general and rigorous as long as x and i remain indeterminate; it would cease to be so if x took on determinate values … . We will later (chap. V) deal with these particular cases and their consequences. [35]

The relevant distinction is that between "indeterminate" and "determinate" values: a function, or, more generally, a variable quantity, is an object of complex nature. A variable quantity, denoted by a letter, *stands for* any possible particular value (potential level), and *may be* given any values (actual level). But, for Lagrange, the use of letters is not a mere shorthand, a way to denote any particular number; there is an *autonomous* level on which indeterminate quantities are to be dealt with, a level whose autonomy is often referred to by bringing up the "generality of algebra" (*la généralité de l'algèbre*). The theorem establishing the generality of the power-series form belongs to this theoretical level, regardless of the actualization (or specialization) of the variable quantity as a number. Yet, this *autonomous* level is by no means an *independent* level; properties proved at the "generality of algebra"-level (let's call it level A) have an implicit counterpart on the "determinate value"-level (level B), as we shall see more clearly by reading the paragraph in which Lagrange deals with his "particular cases". The complete title of Lagrange's Chapter V reads: "On the development of functions when the variable takes on a determinate value. Cases in which the general rule fails to apply. On the values of those fractions whose numerator and denominator vanish simultaneously. On the singular cases in which the development of the function fails to proceed according to the

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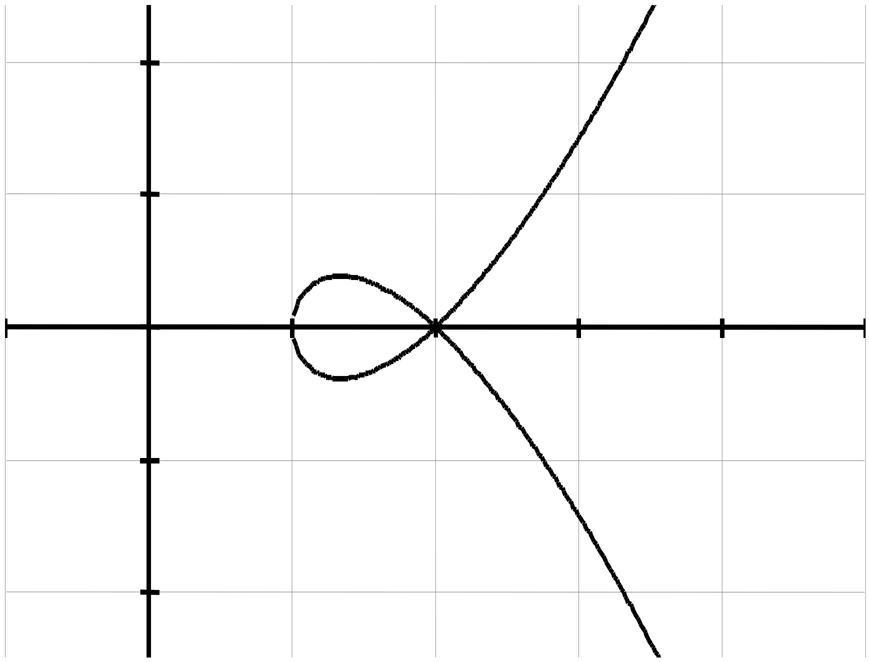
[34] Mais pour ne rien avancer gratuitement, nous commencerons par examiner la forme même de la série qui doit représenter le développement de toute fonction *f*(x) lorsqu'on y substitue $x+i$ à la place de x, et que nous avons supposée ne devoir contenir que des puissances entières et positives de i.
Cette supposition se vérifie en effet par le développement des différentes fonctions connues ; mais personne, que je sache, n'a cherché à le démontrer a priori, ce qui me paraît d'autant plus nécessaire, qu'il y a des cas particuliers où elle ne peut pas avoir lieu. (Lagrange 1813: 22)

[35] Je vais d'abord démontrer que, dans la série résultante du développement de la fonction *f*($x+i$), il ne peut se trouver aucune puissance fractionnaire de i, à moins que l'on ne donne à x des valeurs particulières. … Cette démonstration est générale et rigoureuse tant que x et i demeurent indéterminés ; mais elle cesserait de l'être si l'on donnait à x des valeurs déterminées … . Nous examinerons plus bas (Chap.V) ces cas particuliers et les conséquences qui en résultent. (Lagrange 1813: 22)

positive and whole powers of the increment to the variable." [36]. Let us just quote the first few lines:

> The methods given so far for the development of function $f(x+i)$ rest on the assumption that the form of this development is
> $$f(x)+if'(x)+\frac{i^2}{2}f''(x)+\ldots;$$
> it is thus necessary, before we proceed further, to study why and when this form could fail to appear. We showed earlier (n°2) that it may only be the case when x is given a determinate value which causes some radical to vanish in function $f(x)$ and in all its derivatives. Now, there are only two ways for a radical to vanish, either because the quantity by which the radical is multiplied vanishes, or because the radical itself vanishes. [37]

To explain what he meant, Lagrange used the example of function $f(x) = (x-a)\sqrt{x-b}$. If we draw the curve for $a = 2$ and $b = 1$ (which Lagrange didn't do), we get:



[36] Du développement des fonctions lorsqu'on donne à la variable une valeur déterminée. Cas dans lesquels la règle générale est en défaut. Des valeurs des fractions dont le numérateur et le dénominateur s'évanouissent en même temps. Des cas singuliers où le développement de la fonction ne procède pas selon les puissances positives et entières de l'accroissement de la variable (Lagrange 1813: 57)

[37] Les méthodes que nous venons de donner pour le développement de la fonction f(x+i) supposent que ce développement est de la forme
$$f(x)+if'(x)+\frac{i^2}{2}f''(x)+\ldots;$$
il est donc nécessaire, avant d'aller plus loin, d'examiner quand et comment cette forme pourrait être en défaut.

Nous avons déjà montré plus haut (n°2) que cela ne peut arriver que lorsqu'on donnera à x une valeur déterminée telle qu'elle fasse disparaître dans la fonction f(x) et dans toutes ses dérivées quelques radicaux. Or un radical ne peut disparaître dans une fonction que de deux manières, ou parce que la quantité qui multiplie le radical devient nulle, ou parce que le radical lui-même devient nul. (Lagrange 1813: 57)

One has to remember that for Lagrange, the square root function is two-valued; for instance, 4 has two square roots, 2 and –2, which explains the symmetry of the curve. The function has two singular values, value $x = 2$ for which "the quantity by which the radical is multiplied vanishes" and value $x = 1$ for which "the radical itself vanishes". In this paragraph, Lagrange showed the reader how the development of $f(x+i)$ can be found if $x = 1$ or 2, with power series featuring negative and fractional powers of i: a more general form (meaning: the usual form is a particular case) is needed to deal with singular values, that is to reach universality on the B-level (the extended form is valid for any values of x); the "general rule" was general because it "lived" on the A epistemic level, and[38], because it was generally valid on the B-level, that is for all but singular (i.e. isolated on the straight line and singular for the specific function under study) values of x.

It is also worth commenting upon the generation and use of examples in Lagrange's treatise. Function $f(x) = (x-a)\sqrt{x-b}$ is clearly the simplest case showing both types of singular behaviors: Lagrange didn't set out to chart a closed functional world using point-set properties, but formally generated functions from the most simple elements. In the "rigorous" configuration that we studied in the first part of this chapter, simplicity was irrelevant (though not necessarily ill-considered); here, it is of the essence. The use of examples also differs. The case of $f(x) = (x-a)\sqrt{x-b}$ doesn't support any general statement, its role is pedagogical: it helps the reader spot the potentially singularity-bearing forms, and teaches her how to deal with them. Its simplicity makes it both a paradigm (to be used as a model) and a generic example: in a generically-structured open function-world, dealing with the simplest of all singular cases is the most obvious (if not the only possible) general move; anything that can be said about one of the basic building-blocks is of general interest.

2.2 Two mathematical configurations.

Let us use a table to summarize some of the elements of comparison that we have come across so far, and add a few others as well. For the right column, I chose Camille Jordan as a representative for the "rigorous" configuration in its mature form: his 1890s *Cours d'Analyse* is a standard landmark, in which the works of Dirichlet, Riemann, Weierstrass, Heine, du Bois-Reymond and Dini are reflected. Needless to say this table is just a rough sketch, but rough sketches can prove illuminating at times. This one might not be; we hope it fuels reflection though.

	Lagrange	Jordan
1	Genetic description, bottom-up journey across an open function-world	Axiomatic definition, top-down journey in a closed function-world
2	*Elementary functions* as starting point	*General* functions as starting point
3	Classification of formulae/functions according to *form*, differential diagnosis helping you make out the variety of function *types*	Classification of maps/functions according to (point-set) *properties*, delineation of function *classes*
4	Generality derives from *simplicity* (proper identification of the most simple and elementary forms)	Generality derives from *rigor* (careful wording of hypotheses, counter-example-proof statements, detailed proofs)

[38] Behind this « and » lies the whole dialectic between the two levels. We can but touch here on this fascinating topic.

5	Examples are simple cases which teach you how to deal with whatever you may come across out there	Examples are mainly counter-examples, they both illustrate and motivate the lengthy hypotheses and mind-boggling conceptual distinctions
6	In terms of *genre*, the book is both a *treatise* and a *textbook*.	In terms of *genre*, the book is first and foremost a *treatise*.[39]
7	*Concept image* (for the function concept)	*Concept definition*
8	Convince oneself, convince a friend	Convince an opponent

The first 5 points summarize things said earlier.
In points 7 and 8 we borrow concepts from the didactics of mathematics. The distinction between concept image and concept definition comes from the work of M. Tall and S. Vinner on the psychology of mathematical learning:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kind, changing as the individual meets new stimuli and matures. [40]

The concept image is a much larger and piecemeal cognitive structure than the concept definition (that is, the formal definition, if there is one). To succeed in mathematics, at least in higher education, a student has to display some degree of cognitive flexibility, enabling her to switch from concept image to concept definition in some cases (say, to write down a proof) or the other way round in other cases (say, to devise an easy counter-example to a false statement); in the worst cases, the concept definition is not included in the concept image: the definition may be learnt by rote, still it means nothing to the student. As for the function concept, we saw that hard and fast definitions were rarely found until the second half of the nineteenth century. The few lines of explanation which can be read in the first paragraphs of Euler, Lagrange or Lacroix's treatises are not meant as definitions on which proofs can be based, they're description of what can be called functional dependence, a description that has to be vague enough so as to be *fitted out* to anything that might come up in mathematics and mathematical physics. Lacroix put it bluntly: after his short explanation of the word

[39] The question of *genres*, their historical evolution and epistemological implication, is in itself well-worthy of study; we are not going into that here. We are simply referring to a *growing* tension between "mathematics for professional mathematicians" and mathematics for non-mathematicians (engineers, physicists, maths-teachers or undergraduate students) through the nineteenth century. In his 1893 series of talks on the occasion of the Chicago World's fair, Klein stressed this problem: "Now, just here a practical difficulty presents itself in the teaching of mathematics, let us say of the elements of differential and integral calculus. The teacher is confronted with the problem of harmonizing two opposite and almost contradictory requirements. On the one hand, he has to consider the limited and as yet undeveloped intellectual grasp of his students and the fact that most of them study mathematics mainly with a view to practical applications; on the other, his conscientiousness as a teacher and man of science would seem to compel him to detract in nowise from perfect mathematical rigour and therefore to introduce from the beginning all the refinements and niceties of modern abstract mathematics. … The second edition of the *Cours d'analyse* of Camille Jordan may be regarded as an example of this extreme refinement in laying the foundations of infinitesimal calculus." (Klein 1894: 49).
[40] (Tall and Vinner 1981: 152)

"function", he added: "the use of this word will throw light on its meaning" [41] … pure concept image.
Point 8 comes from Mason, Burton and Stacey's *Thinking mathematically* [42], when reflecting on the function of proof. You can try to write down a proof in order (1) to convince yourself of some mathematical fact (2) to convince a friend or a student, someone of good will but in need of some kind of explanation (3) to convince someone who assumes a systematically skeptical posture, someone who will look for flaws in every step of your reasoning. The proofs in Lacroix and Lagrange are, to some extent, of the first and second kinds. In the 1890s, Jordan's treatise presents a coherent system of subtle definitions and detailed proofs; those proofs and definitions are the result of 70 years of co-evolution of proofs and concepts in mathematical Analysis, an evolution triggered by fierce proof-analysis and counter-example devising[43]. This know-how in proof-design accumulated as a result of the assumption of the "opponent" role by prominent mathematicians, as is exemplified in this passage from one of Abel's letters to his friend Holmboe (1826):

> I doubt you will be able to put forward more than a small number of theorems dealing with infinite series, to the proof of which I can't object with good grounds. Do that, and I shall answer you. [44]

The friendly tone of this letter shows very clearly that this *epistemic posture* has nothing to do with personal enmity or scientific controversy.

2.3 Cauchy's concept of continuity as the answer to a generality challenge.

From a historical point a view, the lectures which Cauchy gave at the *Ecole Polytechnique* in the early 1820s should provide a missing link between the two configurations. On the one hand, they can be analyzed as a globally anti-Lagrangian move, and had a profound influence on the pioneers of the new epistemic style, Abel and Dirichlet. On the other hand, Cauchy shared with Lagrange some basic views as to what functions were, as to the role of singular points etc. which paint a picture of the function world in sharp contrast with what we described in the first part of this chapter. We think the depiction of Cauchy as an in-between figure – in-between two coherent epistemic configurations – helps makes sense of his somewhat puzzling concept of continuity.
We need not expatiate on the first point (the "down with Lagrange" part); quoting the famous introduction to l'*Analyse algébrique* (1821) will suffice:

> As to the methods, I strove to give them the very rigor that is demanded in Geometry, so as to never resort to arguments based on the generality of Algebra. It seems to me that this kind of arguments, though quite commonly acknowledged, most of all when passing from finite to infinite series, and from real to imaginary expressions, can be considered but mere induction; this kind of induction can help sense some truth but is not in keeping with the praised rigor of the mathematical science. It has to be noted that they led us to ascribe an indefinite scope to algebraic formulae whereas, in reality,

[41] L'usage de ce mot en éclairera la signification (Lacroix 1867: 1)
[42] (Mason 1982)
[43] See (Volkert 1987) or the appendices in (Lakatos 1976).
[44] Je crois que tu ne pourras me proposer qu'un très petit nombre de théorèmes contenant des séries infinies, à la démonstration desquels je ne puisse faire des objections bien fondées. Fais cela, et je te répondrai. (Abel 1892: 257)

most of these formulae only hold under certain conditions and for certain values of the quantities which appear. [45]

The « generality of Algebra » is deemed incompatible with mathematical rigor. Cauchy rejected the (implicit) dialectic between levels A and B: level A has but a heuristic value (at best: it proves deceiving, more often than not), level B is the only firm ground on which to base mathematical statements; it has become "reality".

The discarding of level A created a new generality challenge for Cauchy. The didactic genre of the *traité d'analyse* called for a general treatment of all functional situations and for the laying out of a systematic exposition. Lagrange had met these requirements by resorting to a universal (enough) *form* for functions, the power series; obviously, this is not an option for level-A-skeptic Cauchy. We think that two concepts played, for Cauchy, this central role that the power-series form had played for Lagrange: the concept of limit and the concept of continuity. Both strictly operate on the B level, they refer to the numerical behavior of variable quantities. Limiting processes allowed Cauchy to tackle (and to some extent, invent) problems of existence for functions: the exponential function, the primitive and the derivative of a continuous function, the solution of an ordinary differential equation (when regular initial conditions are given) are functions whose existence is proved by a limiting process. A short comparison with Lacroix will help illustrate this point, in the case of primitives. After presenting the (formal) rules for differentiating functions, Lacroix wrote that finding the primitive (or indefinite integral) $\int f(x)dx$ of a function $f(x)$ is the reverse problem to that of finding the derivative; he then presented various methods to (formally) solve this problem; the definite integral $\int_a^b f(x)dx$ was then introduced, and various numerical methods were presented to help find an approximate value for this definite integral, in case no primitive could be formally obtained. Cauchy proceeded exactly the other way round: he used the approximation methods to prove that the symbol $\int_a^b f(x)dx$ stood for a well-defined number (provided f is continuous between a and b), then allowed quantity b to vary, thus defining a new numerical function which was proved to have $f(x)$ as its derivative.

Cauchy also had to distinguish between regular values and singular values. Lagrange had done it by analyzing the form of a function; this, again, is not an option for Cauchy. This is where, in our view, the concept of continuity comes into play. To the (cautious) modern reader, Cauchy's continuity concept is a cause for puzzlement. One the one hand, his definition looks like our numerical, point-set theoretic definition. On the other hand, Cauchy used this definition in ways which the modern reader finds either inconclusive (for lack of distinction between continuity and uniform continuity, for instance) or altogether misleading; for instance, Cauchy "established" that the limit of a sequence of continuous functions is a continuous function … a fact to which counter-examples were known by the time of Cauchy ! Another puzzling feature is that for Cauchy, continuity was always assumed to hold in an

[45] Quant aux méthodes, j'ai cherché à leur donner toute la rigueur qu'on exige en géométrie, de manière à ne jamais recourir aux raisons tirées de la généralité de l'algèbre. Les raisons de cette espèce, quoique assez communément admises, surtout dans le passage des séries convergentes aux séries divergentes, et des quantités réelles aux expressions imaginaires, ne peuvent être considérées, ce me semble, que comme des inductions propres à faire pressentir quelquefois la vérité, mais qui s'accorde peu avec l'exactitude tant vantée des sciences mathématiques. On doit même observer qu'elles tendent à faire attribuer aux formules algébriques une étendue indéfinie, tandis que, dans la réalité, la plupart de ces formules subsistent uniquement sous certaines conditions, et pour certaines valeurs des quantités qu'elles renferment. (Cauchy 1989: ij)

interval (of non-null length) and discontinuity assumed to occur at isolated points, in spite of the fact that the Cauchy definition seemed to allow for functions with dense discontinuity loci (as in Dirichlet) and functions which were continuous at isolated points only (though those didn't come up until much later).
I think questions of generality help understand this baffling "continuity" concept, in a twofold way. First, Cauchy's use of his continuity concept is understandable when one refers to its epistemic role instead of focusing on the numerical definition. Cauchy's continuity hypotheses serve him right as hypotheses of general/regular behavior: to some extent, what matters is not what Cauchy *means* when he writes "continuous", but what he *means to do*. The second element is not specific to Cauchy but plays a part in Cauchy's choice of the continuity concept as generality-bearing concept. It can also be found, for instance, in Ampère's famous proof of the "fact" that continuous functions admit a derivative. A close look at the proof shows that what Ampère meant is that continuous functions have a derivative, *save for isolated points*. In every step of the proof, Ampère acknowledged the fact that for specific values of the variable, the general behavior that is aimed for may not hold … yet he didn't mention it when stating the final theorem. I believe this unwritten rule that "when dealing with functions, all that is stated and proved is so, *save for, maybe, isolated values of the variable*" is an implicit but essential part of a mathematical configuration which is common to Lagrange, Ampère *and* Cauchy. Singular (that is, for Cauchy, "discontinuity") points may drop out of sight, and are (implicitly) assumed to appear only as isolated points. Formally universal statements are to be read modulo this proviso, this is the price to pay for a general statement, that is, one that deals with all functions, whatever they may be [46].
To sum up, in spite of radically opposing views as to the legitimacy of the A-level, it seems to us that Lagrange and Cauchy had at least this in common: they studied the *general* behavior (that is, except for isolated values of the variable) of what would later be termed "usual functions". The numerical setting chosen by Cauchy paved the way for the monster-making business, but it is not a business in which Cauchy engaged, or even a business which he considered; the functions which Cauchy studied were the same analytic function which Lagrange studied and the very same properties were to be attained, though in a completely different way. A systematic comparison between Lagrange and Cauchy helps identify similar *generality demands*, and stresses the *functional equivalence* between the form of the power series (in Lagrange) and the general property of continuity (in Cauchy). Both mathematicians faced two generality demands: one that pertained to the *genre* "general treatise" and called for the identification of a unifying element (be it form or property); one that was more content-specific and derives from the fact that functions operate (partially or exclusively) on the B-level, which called for a distinction between intervals of regular behavior (where general proven statements hold) and isolated irregular points.

3. Logical generality *vs embedded* generality: Cauchy vindicated.

We would eventually like to point to a third interaction between questions of generality and the development of function theory in the nineteenth century. On this occasion, we shall introduce the concept of *embedded* generality, in order to document both the reflexive character of mathematics and a specific form of general statement. As for the term "logical generality" against which I shall contrast "embedded generality", it is taken from Poincaré (see below): it simply denotes the standard idea that a case is more general than another one if the first one extensionally encompasses the second one.

[46] See chapter 4 in (Chorlay 2007)

3.1 A “small corner” for “good” functions ?

In a 1904 paper on definitions in mathematics, Poincaré took a backward look at the development of rigor in mathematical Analysis over the nineteenth century; the mood was significantly different from that of the quotation which we gave in the introduction to this paper:

> Logic sometimes begets monsters. In the last half-century, we saw the emergence of a bunch of bizarre functions, the purpose of which seems to be to differ as much as possible from these straightforward functions [*honnêtes fonctions*] that prove useful. No more continuity, or continuity without derivability etc. What’s more, from the logical point of view, these weird functions [*fonctions étranges*] are the most general, those that we came across without looking for them now appear to be but a particular case. They are left with but a small corner. In the old days, when a new function was invented, it was for a practical purpose; nowadays, they are invented for the very purpose of finding fault in our fathers’ reasoning, and nothing more will come out of it. [47]

Rigor developed, Poincaré lamented, at the cost of fruitfulness; in his view, the “straightforward functions” should have remained the main topic of study, and younger mathematicians spent too much time reveling in the minutiae of the general theory of functions. Yet he could but acknowledge the fact that general functions are more general from a logical viewpoint: they form the all-encompassing class and many subclasses, sub-subclasses, sub-sub-subclasses … can be made out before that of analytic functions, a situation for which Poincaré used the metaphor of the “small corner” (*petit coin*) of the function world.

3.2 The return of Cauchy.

At the very same time however, some mathematicians started using the sophisticated tools of general function theory (and the point-set theory it gave rise to) in order to vindicate the classical (say, Cauchy) point of view. They would use generality arguments to show that the “small corner” is actually large *enough*. Emile Borel’s work is a good example and I will focus on this case. His overall view is put in the clearest of ways in his 1912 analysis of his mathematical work; the following quotation is pretty lengthy, but we feel its skilful weaving of the various threads that we have been following makes it well worth reading:

> There were, there still are, mathematicians who choose to ignore what they deem to be refined subtleties with no practical use; this attitude is indeed legitimate since it leads to results but it seemed to me that I could not stick with it, for several reasons: one the one hand, until now, no one could draw a clear line between *straightforward* and *bizarre* functions; when studying the first, you can never be certain you will not come across the others; thus they need to be known, if only to be able to rule them out.

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[47] La logique parfois engendre des monstres. Depuis un demi-siècle on a vu surgir une foule de fonctions bizarres qui semblent s’efforcer de ressembler aussi peu que possible aux honnêtes fonctions qui servent à quelque chose. Plus de continuité, ou bien de la continuité, mais pas de dérivées, etc. Bien plus, au point de vue logique, ce sont ces fonctions étranges qui sont les plus générales, celles qu’on rencontre sans les avoir cherchées n’apparaissent plus que comme un cas particulier. Il ne leur reste qu’un tout petit coin. Autrefois, quand on inventait une fonction nouvelle, c’était en vue de quelque but pratique ; aujourd’hui, on les invente tout exprès pour mettre en défaut les raisonnements de nos pères, et on n’en tirera que cela. (Poincaré 1904: 263)

On the other hand, one cannot decide, from the outset, to ignore the wealth of works by outstanding geometers; these works have to be studied before they can be criticized. … To my knowledge, Cauchy never explicitly explained what he meant by "function"; a reading of his work seems to me to reveal evidence that, for him, the question didn't arise; "function" was but the general term used to denote any of the particular functions which the analysts study, each of these particular functions had its own definition based on elementary functions (by means of series, integrals, differential equations etc.); it was assumed that any argument pertaining to the general "function" would apply to all particular functions which would later be discovered, provided they meet the conditions appearing in the propositions (most of the time, these conditions are continuity for the function and its derivative).

In the very same way, a biologist would refer to "living beings" or a chemist to "simple elements" without having had to delineate an *a priori* concept of the living being *per se*, or the simple element *per se*; they simply have in mind the living beings that they know or could know of.

This Cauchy viewpoint was contrasted with the seemingly more general method in which one starts with a function given *a priori* as a correspondence which can be devised regardless of explicit formulation; … here is no place to discuss whether what can not be formulated can or cannot be an object for science; two remarks will suffice; on the one hand, this more general conception of functions led to the devising and studying of new functions, which would otherwise not have been thought of; thus it proved useful; but, on the other hand, the actual display of analytical expressions representing the newly devised functions made the *a priori* conception useless; after a detour, one comes in fact back to Cauchy's viewpoint.

… My work on divergent series as well as those on monogenous functions can be traced directly to Cauchy's ideas; in these works just as well, I used the improvements to the rigor of analysis worked out by Cauchy's successors, while breaking free from the too narrow conceptions which they introduced along with that very rigor. [48]

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[48] Il y a eu, il y a encore des mathématiciens qui ont pris le parti d'ignorer ce qu'ils considèrent comme des raffinements de subtilité sans portée pratique ; cette attitude est assurément légitime, dans la mesure où elle conduit à des résultats, mais il ne m'a pas semblé possible de m'y tenir, pour plusieurs raisons: d'une part, jusqu'ici, nul n'a indiqué une démarcation nette entre les fonctions honnêtes et les fonctions bizarres ; lorsqu'on étudie les premières, on n'est jamais sûr de ne pas voir apparaître les secondes ; il faut donc les connaître, ne fût-ce que pour savoir les exclure. D'autre part, on ne peut pas ignorer, de parti pris, un ensemble considérable de travaux dus à des géomètres éminents ; on doit les étudier avant de les critiquer.

… Cauchy n'a jamais, à ma connaissance, exposé explicitement ce qu'il entendait par une fonction ; la lecture de son œuvre me paraît montrer avec évidence que, pour lui, cette question ne se posait pas ; « fonction » était simplement le terme général qu'il employait pour désigner l'une quelconque des fonctions particulières considérées par les analystes, chacune de ces fonctions particulière ayant sa définition propre, à partir des fonctions élémentaires (au moyen de séries, d'intégrales, d'équations différentielles, etc.) ; il est sous-entendu que les raisonnements faits sur la « fonction » en général s'appliqueront, en outre, aux fonctions particulières qui pourront être découvertes ultérieurement et qui possèderont les propriétés spécifiées dans les énoncés (propriétés qui consistent, le plus souvent chez Cauchy, en la continuité de la fonction et de sa dérivée).

C'est ainsi qu'un biologiste peut parler d'« être vivant », ou un chimiste d'un « corps simple » sans avoir été obligé de créer une conception a priori de l'être vivant en soi ou du corps

Two significant examples will help understand Borel's subtle stand. To understand the first example, one must recall that in Lagrange, Lacroix or Cauchy, singular points were always assumed to be isolated (on the straight line); in higher dimensions, the locus of singular points was always assumed to be of non-null codimension in the parameter space, which is why statements such as "a real-valued square matrix is, generally speaking, invertible" or "three points in a plane aren't usually aligned" could be given precise mathematical meaning (which made them not only meaningful but also true !). In the next phase, the displaying of functions whose locus of singular points was not made of isolated points was one of the most active industry in the monster-making business, and Weierstrass' everywhere singular continuous function was the monster *par excellence*. Borel went one step further, so as to ascertain the generality of the most straightforward of all functions, namely polynomial functions: *y* being a bounded function, defined over the 0-1 interval,

> Given two positive and arbitrarily small numbers ε and ε', one can determine a polynomial P(*x*) such that the points at which *y*-P(*x*) is, in absolute value, greater than ε make up a set of measure less than ε'. … One can also say that, by letting ε and ε' tend to zero, there is a sequence of polynomials that tend towards *y*, except at the point of a set of null measure.
>
> This result is essential for the theory of functions of one real variable, since it shows that the singularities of such a function *fill very little room*; it is thus possible, in many circumstances, to proceed as if they didn't exist. The in depth study of the notion of a set of null measure thereby leads to an middle stand between these geometers who are inclined to consider only "good" functions and those who could be led to think that "good" functions are but an extremely particular case. We know, in a precise way, that neither party have it completely wrong. [49]

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simple en soi ; ils pensent simplement aux être vivants qu'ils connaissent ou qu'ils pourraient connaître.

On a opposé à cette manière de voir de Cauchy la méthode, en apparence plus générale, qui consiste à se donner la fonction a priori, comme une correspondance qui n'a pas besoin d'être formulée explicitement pour être conçue ; … ce n'est pas ici le lieu de discuter si ce qui ne peut pas être formulé peut être réellement objet de science ; deux remarques nous suffiront ; d'une part, cette conception plus générale de la fonction a conduit à construire et à étudier de nouvelles fonctions auxquelles on n'eût, sans doute, pas songé sans elle ; elle a donc été utile ; mais, d'autre part, cette construction effective d'expression analytique représentant les fonctions conçues a eu pour résultat de rendre désormais inutile la conception a priori de la fonction ; après un détour, on revient, en fait, au point de vue de Cauchy.

… Mes travaux sur les séries divergentes, comme ceux sur les fonctions monogènes, se rattachent directement aux idées de Cauchy ; là aussi, j'ai utilisé les perfectionnement apportés à la rigueur de l'analyse par les successeurs de Cauchy, mais j'ai su me dégager des conceptions trop étroites introduites en même temps que cette rigueur. (Borel 1972: 120)

[49] Etant donnés deux nombres positifs arbitrairement petits ε et ε', on peut déterminer un polynôme P(*x*) tel que les points où la différence *y*-P(*x*) est plus grande en valeur absolue que ε forment un ensemble de mesure inférieure à ε'. … On peut aussi dire, en faisant tendre ε et ε' vers zéro, qu'il y a une suite de polynômes qui tend vers *y*, sauf pour les points d'un ensemble de mesure nulle.

Ce résultat est essentiel pour la théorie des fonctions d'une variable réelle, car il montre que les singularités de ces fonctions *occupent très peu de place* ; il est par suite possible, dans bien des circonstances, de procéder comme si elles n'existaient pas. On est ainsi conduit, par l'étude approfondie de la notion d'ensemble de mesure nulle, à prendre une position en

From a logical point of view, the concept of continuous function is much more general than that of a polynomial function; but the refined tools of point-set theory help us go beyond this simple fact, they help us asses just *how much* more general they are. And it turns out that, in some way, polynomial functions are general *enough.* This kind of generality is context-dependent, in two ways. It depends on the mathematical tool with which one assesses generality (here, the measure of a set of points on the straight line); in this case the polynomial case proves general enough to found the integration theory of continuous functions (goal-dependence):

> The integral of *y* may be defined as the limit of the integrals of polynomials P(*x*). … This example shows how the notion of measure allows us to rid the theory of real functions of much of the complication which had emerged in the logical development of analysis. [50]

The other example comes from Borel's work on power series in one complex variable. In a 1896 note to the *Comptes Rendus de l'Académie des Sciences de Paris*, Borel had given a mere heuristic argument to support the claim that a function defined by a power series whose radius of convergence equals one cannot generally be analytically extended beyond this convergence disc. By 1912, he had devised a more rigorous and much more sophisticated argument, by applying the notion of set of zero-measure *in a function set*. Rephrasing in terms of probability (zero-measure sets are those of zero probability), he summarized:

> I proved that for such a series, picked at random (words whose meaning I made precise), its convergence circle is generally a cut, which means that the cases in which analytic continuation is possible are to be deemed exceptional. [51]

In this example, the logical viewpoint says nothing more than: not all power series (whose radius of convergence equals one) can be analytically extended; the class of analytic functions whose maximal domain of analyticity is the unit disc is strictly included in the class of analytic functions whose maximal domain of analyticity contains the unit disc … Borel goes beyond this rather trivial statement (which could easily be proved by displaying just *one* "bizarre" power-series) and endeavors to assess *how much more general* the second class is. It turns out that, if the mathematical tool used to compare degrees of generality is a measure-theoretic tool of a function-space, the smaller of the two class is so bulky that its complement has probability (or area, to state it more geometrically) zero: in this case, monsters (non-continuable function elements) are the rule and "good" power series are the exception. This statement may sound rather anti-Cauchy, but Borel quite dramatically turns it into a pro-Cauchy argument:

> It is thus illusory to consider Taylor series *a priori*, regardless of its origin, this abstract study can only lead to negative answers. [52]

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quelque sorte intermédiaire entre les géomètres disposés à ne considérer que les « bonnes » fonctions et ceux qui auraient pu être tentés de croire que ces « bonnes » fonctions ne sont qu'un cas extrêmement particulier. Nous savons d'une manière précise que ni les uns ni les autres n'ont tout à fait tort. (Borel 1972: 122)

[50] L'intégrale de *y* peut être définie comme la limite des intégrales des polynômes P(*x*). … Cet exemple montre comment la notion de mesure permet de débarrasser la théorie des fonctions de variable réelle de la plupart des complications qui y avaient été introduites par le développement logique de l'analyse. (Borel 1972: 123)

[51] J'ai démontré qu'une telle série choisie au hasard (mots dont j'ai précisé le sens) admet en général son cercle de convergence comme coupure, c'est-à-dire que les cas où le prolongement analytique est possible doivent être regardés comme exceptionnels. (Borel 1972: 123)

Thus, not only the abstract, *a priori*, all-encompassing definition of a function as a correspondence is of little worth, but even in the realm of analytic functions, Weierstrass' abstract, *a priori* definition based on the notions of power series and analytic continuation fails to delineate the class of "good" functions, preposterously so.

3.3.1 Embedded *generality*.

In the first part of this chapter, we showed how the search for a more rigorous and systematic function theory – one that could encompassing the difficult case of functions studied with Fourier series – resulted in significant changes in the function concept and in the adoption of new systematic ways of charting the function world. Essential features of this new configuration are those which Poincaré or Borel call "logical": the use of an abstract definition of a function, the delineation of function classes (or function sets) by abstract characteristic properties and milestone standard examples. In this viewpoint, if a function class C_1 is strictly included in function class C_2 (extensional side of the logical viewpoint), then C_2 is more general than C_1 … and that's about it.
Yet, as we saw with Borel's example, this extensional viewpoint can serve as stepping stone for a new kind of investigation: C_2 may be logically more general than C_1, but *how much so* ? Can't C_1 be *general enough* for some purpose ? Is C_1 so special that it can, in some circumstances, be *neglected* altogether ? Those questions are by no means specific to function sets or function theory; any set of objects, any parameter space for some mathematical situation can be investigated in this way. The tools with which the degree (or relative degree) of generality is assessed is a mathematical tool, though, in most cases, not a number (as the word "degree" might suggest). To compare in terms of "size" two sets, one being part of the other, a wealth of methods is available to the twentieth-century mathematician. Dimensional arguments had been in use since classical mathematics: a doubly-infinite set is significantly larger than a simply-infinite one, though the fact that the smaller one may disconnect the larger one may give it a global topological importance that its "size", alone, doesn't account for; in this respect, it is safer to neglect a subset of singular cases whose dimension is at least two degrees lower than that of the space of all cases. With the advent of point-set theory in the last years of the nineteenth century, a great variety of new tools were made available. Let us give but a few simple and context-free examples. Consider the set C_1 of positive rational numbers less than 1 and the set C_2 of positive real numbers less than one. One of the mathematical tools that can be used is that of density: C_1 is dense in C_2, which (loosely speaking) means that there are elements of C_1 "everywhere" in C_2. In some cases, it makes C_1 general enough; for instance, to check that two real-valued continuous functions f and g are equal on C_2, it suffices to check that they are equal on C_1. Another mathematical tool is measure theory. Let's say that the probability (or measure) of an interval (which is a part of C_2) is its length, and that the probability of a (denumerable) infinity of pairwise disjoint intervals is the sum of their probabilities/lengths, then C_1 has probability (or measure) 0; C_1 seems to be completely negligible compared to C_2 (which has probability 1): if a number is chosen at random in C_2, the probability that a C_1 number be chosen is 0. As Borel remarked, this is relevant for integration theory. We wish to coin the term "*embedded* generality" for this kind of generality assessment which relies of the description of a mathematical structure (whether of set, ordered set, measured space, topological space, manifold etc.) on a set of objects or parameter space for mathematical situations. For instance, the search for the right structure in the case of the qualitative theory of dynamical systems is beautifully illustrated in

[52] Il est donc illusoire de considérer la série de Taylor *a priori*, indépendamment de son origine, cette étude abstraite ne peut conduire qu'à des résultats négatifs. (Borel 1972: 124)

Tatiana Roque's chapter in this volume. One of the striking features of *embedded* generality is its *twofold context-dependence*: dependence on the purpose and on the measuring-tool. In our simple example, C_1 can turn out to be either general enough or completely negligible. This concept testifies to the reflexive nature of mathematics, its ability to turn apparently (and formerly) *meta*-level questions *about* mathematics (such as: the comparison between two theories, the degree of generality of a class of objects/statements, the choice of the class of objects which are really worth investigating) into mathematical questions, by designing the proper mathematical tools (e.g. group relations to study the relationships between various geometrical theories, assessment of *embedded* generality).

We came across this concept of *embedded* generality in our discussion of the interactions between questions of generality and the development of function theory in the nineteenth century, but we do not mean to say it emerged in this context. For instance, Anne Robadey's chapter documents Poincaré's devising of measure-theoretic arguments in celestial mechanics; her historically detailed and epistemologically informed narrative shows how Poincaré managed to turn a loosely-formulated corollary to a (false) theorem into a full-fledged rigorous theorem about the general behavior of orbits, by describing the parameter space of orbits with tools he imported from the theory of continuous probability. He thus kicked off the theory of dynamical systems, a theory in which several types of embedded generality arguments are of the essence: Tatiana Roque's chapter on genericity documents at least two generation of such arguments since World War II. Other examples could be found in Poincaré's work, for instance in his work on the so-called "Fuchsian" functions (1881-1885). Presenting the mathematical details would take us far beyond the scope of this chapter, it suffices to know that this example documents the passage from dimensional arguments to topological arguments: to show that two parameter spaces could be identified, Poincaré had to show that they not only were of the same dimension, but also topologically equivalent (homeomorphic). The proof-method he devised on this occasion, the "method of continuity" (*méthode de continuité* in French, *Kontinuitätsbeweis* in German), would stir admiration (and disbelief) until the 1920s [53].

Conclusion.

The content of this chapter emerged from an exploration of the use of the word "general" in a well-known corpus, that on the foundation of function theory in the nineteenth century. In keeping with the spirit of this handbook, we endeavored to make sense of the wealth and diversity of occurrences by focusing on topics such as the use of examples, exceptions and singular cases; by focusing, also, on ways of expressing and assessing generality. These guiding threads led us to identify three distinct configurations which we strove to characterize. As descriptive terms, we used both *epistemic configuration* and *mathematical practice*: the first referred to closed (at least coherent) epistemic structures (with their own rules for action), the second referred to the way in which epistemic configurations were dealt with by mathematicians (in accordance to or, at times, in spite of the rules).

These three configurations are by no means independent, quite the contrary; we clearly opted for a kind of dialectical narrative, in which the third phase was explicitly described – sometimes by mathematicians themselves, such as Borel – as a synthesis of the former two. In a sense, this dialectic movement relies not so much on three concepts of generality but more on a feature that is specific to the objects under study: mathematical functions. Indeed, a function is a two-faced entity: it can either be considered as an *individuum* – when a formula is written down, or when the function is proved to be an element of a given class of functions

[53] See, for instance, chapter 5 in (Chorlay 2007).

– or as a *dynamic plurality* – as a correspondence between numerical values; in the latter viewpoint, the behavior of a given function can, in turn, be considered general for some numerical values of the variable and singular for some other values. This Russian doll structure accounts for much of the complexity of the story we tried to tell. On the basis of our analysis of the third phase – in which tools from point-set theory first designed to describe the singular sets of values of an arbitrary function started to be used to distinguish among functions in function sets – we eventually endeavored to define the concept of *embedded* generality, which we think is specific to the mathematical (or mathematicised) sciences but not to mathematical Analysis.

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