Category Theory and Philosophy of Mathematics

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Our Journey

The paper is about the hypothesis of a ‘categorical turn’ in 1) Mathematics ; 2) Logic ; 3) Philosophy of Mathematics.

Is analytic philosophy sensitive to the notion of ‘turns’ ? It may seem that it is not, because analytic philosophers never learnt about Heidegger’s *Kehre*.

Still, analytic philosophy is part of the so called ‘linguistic turn’ of 20^{th} century’s philosophy.

Also, it seems that analytic philosophy is open to the idea that some scientific event may change the face of thought and philosophy. Is not invention of contemporary first order logic (from Frege to Tarski) described in that way ? Did not analytic philosophers argue time and time again that after Riemanian geometry, Kant's transcendental aesthetics could not be sustained anymore ?

I guess the issue of a possible 'categorical turn' is an interesting one, for any kind of philosopher.

The mathematical event

In *Categories for the Working Mathematician*, S. MacLane makes six remarks about the origin of category theory (invented by him and Samuel Eilenberg) :

1. It was meant for algebraic topology.
2. The idea was to make abstract speculation efficient.
3. It was fun to steal words from philosophers (*categories* from Aristotle and Kant, *functor* from Carnap).
4. Category theory was firstly understood as a new language.
5. The basic vision was that their morphisms are essential to objects (a vision MacLane traces back to Emmy Noether).
6. Ehresman was the most radical with respect to that vision, as he went as far as exposing categories as the algebraic system of their morphisms.

As a first approach to category theory, I shall give an approximative quote taken from an introductory course I read very long ago (written by Pierre Lusson) : "(. . .) the reader will easily reach the conviction that, with respect to mathematics, categories are like clarinet with respect to New Orleans jazz".

What Lusson meant is that once we know the definition of categories, we immediately see categories associated with any particular field in mathematics (category of groups, category of topological spaces, category of ordered sets).

In other words, concept of category seems to replace the heavy and unpleasant Bourbaki notion of 'espèce de structure'.

Still, categories are or seem to be 'indifferent' to the size of concerned region of the universe of sets. They may be a strict class, too big to be a set, like in the case of groups or topological spaces, but categories may also spontaneously inhabit some set (like the category associated with an ordered set).

Categories do more than purely descriptive job. We come to realize that some facts may be proved at the general level of category theory : Snake lemma or Yoneda lemma are standard examples, appearing early in textbooks.

Categorical technique is connected with important contemporary developments, such as algebraic topology or algebraic geometry. It plays its part when fundamental 'dictionaries' are introduced : we 'read' a topological space through associated groups, or we build a topological space out of a ring, and category theory describes what we do and what we get. Not only category theory proves new general results, it also collaborates at new generative procedures.

And eventually, it provides new interpretations of notions at stake in mathematical tradition : it is for example possible to regard what has been classically understood as 'geometrical situation' as given by a sheaf of algebraic objects over a topological space.

Category theory also allows us to 'externalize' mathematical notions : something connected with the fundamental attitude of viewing objects in terms of their morphisms. As the most evident case, we may mention 'points'. In the category of sets, we observe that points of a set A may be seen as maps $\{\star\} \xrightarrow{f} A$ [$a \in A$ is identified with map sending $\star$ into a]. This can be generalized at the level of an arbitrary category, provided there is a terminal object 1. Points of an object A will be defined as arrows $1 \xrightarrow{p} A$.

"External" categorical constructions may sometimes be relocated inside some set. Let me take the example of the Brauer group $Br(k)$ of a field k. It is meant as a classifying space for central simple finite dimensional algebras over k : finite dimensional vector spaces A over k on which a multiplication is also defined, making A a "ring" for that operation and vector's addition.

Algebra A is *central* if the only elements in A commuting with every element are the $\lambda.1_A$ for λ in k. It is *simple* if the only proper ideal for ring structure is $\{0_A\}$. Standard examples are matrix algebras $\mathcal{M}_n(k)$.

We get an equivalence relations on such algebras by defining $A \simeq A'$ iff there exist n and n' such that $A \otimes \mathcal{M}_n(k)$ is isomorphic to $A' \otimes \mathcal{M}_{n'}(k)$. Then, what we would like to do is to consider equivalence classes for that relation, and define an "addition" for such classes, by making $Cl(A) + Cl(B)$ simply $Cl(A \otimes B)$ [which entails that we have to verify that $A \otimes B$ is also central simple when A and B are, and that our definition is stable with respect to $\simeq$ relation].

Unfortunately, it seems that there are too many finite dimensional central simple algebras, bringing us beyond the 'set level'. Still, difficulty may easily be overcome : we just have to choose some infinite dimensional vector space V over k, and only consider classes defined by algebras which are subspaces of V.

Clearly any class is already represented by some algebra subspace of V. Clearly also, resulting group structure does not depend on the choice of V. This makes the definition of $Br(k)$ legitimate. I have just reproduced the issue as Claude Chevalley taught it very long ago.

Sometimes structural light brought by category theory may seem too strong or too powerful. I will take here the example of an exercise of MacLane's treatise which has surprised me, and to some extent troubled.

It is about the notion of *monad*. A monad is given by a quadruple $(\mathcal{C}, T, \mu, \eta)$, where $\mathcal{C}$ is a category, T an endo-functor of $\mathcal{C}$ (a functor $T : \mathcal{C} \to \mathcal{C}$), μ a natural transformation $\mu : T^2 \to T$ and η a natural transformation $\eta : 1_{\mathcal{C}} \to T$, where $1_{\mathcal{C}}$ stands for the identity functor on $\mathcal{C}$. Furthermore, it is required that following diagrams commute (next slide).

Fundamental properties of a *monad*

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
T \arrow[r, "T\eta"] \arrow[dr, "1_T"'] & T^2 \arrow[d] & T \arrow[l, "\eta T"'] \arrow[dl, "1_T"] \\
& T &
\end{tikzcd}$$

FIGURE: Commutative Diagrams

Here, we should explain what natural transformations $T\mu$, μT, $T\eta$ and ηT are. I prefer to jump quicker to what I have in mind. In order to understand what our three commuting diagrams mean, the easiest way seems to be to consider a typical example of monad structure.

If G is a group, we may associate with it an endo-functor of category Set : functor T associating to any set A the set $G \times A$, and to any map $f : A \to B$ the map $T(f) : (g, a) \in G \times A \mapsto (g, f(a)) \in G \times B$.

In order to get a monad structure, we have to define natural transformations $\mu : T^2 \to T$ and $\eta : 1_{\mathcal{C}} \to T$. μ is known through maps $\mu(A)$ (for a set A), having as source $T(T(A) = G \times (G \times A)$ and as target $T(A) = G \times A$.

We simply put $\mu(A) : (g', (g, a)) \mapsto (g'g, a)$, taking advantage of group multiplication on G. We verify easily that this indeed defines a natural transformation.

For η, we need maps $\eta(A)$ going from A to $G \times A$. We simply choose $\eta(A) : a \mapsto (e_G, a)$, using neutral element e_G of our group. Again, this indeed defines a natural transformation.

It appears then than we have a monad structure : that our left square commutes corresponds to associativity of the group law in G, and that the two right triangles commute corresponds to neutral element property of e_G.

We are not at the end of our story. Textbooks also introduce the notion of *algebra* over a monad $(\mathcal{C}, T, \mu, \eta)$. Such an algebra is based on a specific object A of $\mathcal{C}$, "algebra structure" being supposed to be given by an arrow $h : T(A) \to A$, provided that following diagrams commute (next slide)

Fundamental properties of an algebra over a monad

$$\begin{tikzcd} T^2(A) \arrow[r, "Th"] \arrow[d, "\mu(A)"'] & T(A) \arrow[d, "h"] \\ T(A) \arrow[r, "h"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta(A)"] \arrow[dr, "1_A"'] & T(A) \arrow[d, "h"] \\ & A \end{tikzcd}$$

FIGURE: Diagrams for Algebras

In our typical case of functor $A \mapsto G \times A$ in Set, an algebra structure concerning a set A appears to be a map $h : G \times A \to A$, something we may think of as an external law making G "act" upon A.

Indeed, we may state that required commutation of the left square simply means that $g'.(g.a) = (g'g).a$ in general [using mark '.' for our external law]. In the same vein, commutation of the right triangle simply says that we have $e_G.a = a$ in general. Thus we recover the classical notion of group operating on a set.

This provides an understanding of what monads and algebras are. It seems that a monad structure brings the underlying category some 'algebraic functioning' at the functorial level. In that way it involves as it were our category into algebraic structure. When we have an algebra over our monad, we can say that such involvement gets applied to some object A of $\mathcal{C}$. Algebraic functioning comes to concern specifically object A.

Now we are arriving at what has surprised me. In an exercise, MacLane asks us to recognize as a monad a structure associated to a quite general functor acting on category *Set* : functor T associating to any set A its Powerset $\mathcal{P}(A)$. $T(f)$ is defined, for a map $f : A \to B$, as the 'direct image' function, associating to subset C of A subset $f(C) = \{f(x) | x \in C\}$ of B.

In order to have a monad structure, we then need arrows μ and η. To define transformation μ, we have to decide what maps $\mu(A) : \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A)$ will be. We put $\mu(A)(C) = \bigcup_{B \in C} B$.

And to define η, we have to settle what maps $\eta(A) : A \to P(A)$ are : we put $\eta(A)(a) = \{a\}$, associating to each point its singleton.

We then verify that we have a monad structure : diagrams expected to commute do indeed commute. We may thus consider eventual algebras over our monad. An algebra based on set A will be given by an arrow $h : \mathcal{P}(A) \to A$.

It immediately appears that triangle commutation in the definition entails that for every h yielding an algebra, for every $a \in A$ we must have $h(\{a\}) = a$.

It then happens that if we define $x \leq y$ (in A) by $h(\{x, y\}) = y$, square commutation leads to proving that relation $\leq$ is indeed an order relation. As a matter of fact we prove that according to that relation $h(B) = Sup(B)$ for an arbitrary subset B of A.

The 'shocking' conclusion is that algebras for our monad structure are exactly semi-complete lattices (lattices such that every subset has a *g.l.b.*). So it turns out that objects involved into algebraicity brought by a monad are in that case order structure after all : this seems to "mix" two of the three fundamental 'espèces de structure' of Bourbaki.

I shall use my surprise further in the talk, but for now we may move to second section, concerning logic.

Logic and Categories

The general idea is that category theory has shown a remarkable ability to welcome, reflect or translate logical facts, theories, languages. I shall give a few examples of that, while being sure that there are many others which I simply ignore.

Algebra of Sub-Objects

A first way of approaching connection between logic and categories is to speak about the lattice of sub-objects of a given object in a topos. We are familiar with the boolean lattice $\mathcal{P}(A)$ for A a set (order relation given by $\subseteq$). And we know the correspondence of such lattice structure with logic : $\cup$ goes with $\vee$, $\cap$ with $\wedge$ and $B \mapsto \mathcal{C}_A(B)$ with $\neg$.

The point is that one can transpose that setting in the context of a topos. Subobjects of an object A will here be defined by monomorphisms $B \xrightarrow{i} A$ (only we identify monomorphisms deriving from each other through some isomorphism relating their sources). We then manage to define operations corresponding to $\cup$ and $\cap$.

With such definitions, $Sub_{\mathcal{C}}(A)$ happens to enjoy lattice structure, like $\mathcal{P}(A)$ was. Connection with logic is not lost either. From the source-example, we may also replicate "characters" associated with sub-objects.

In set-theoretic context, the character of subset B of A is function $\chi_B : A \to \{0,1\}$ taking value 1 on B and 0 outside of B, where $\{0,1\}$ is the set or truth values, 0 standing for *false* and 1 for *true*.

In a topos, an analogon of such truth values is given, thanks to the 'subobject classifier' Ω. We may therefore define characters, which keep on characterizing sub-objects, as in set-theoretic case.

And we may even introduce specific arrows doing the job of logical operations, to the effect that we will have again $\chi_{B\cap C} = \cap \circ \langle \chi_B, \chi_C \rangle$ and $\chi_{B\cup C} = \cup \circ \langle \chi_B, \chi_C \rangle$, for arrows $\cap$ and $\cup$ going from $\Omega \times \Omega$ to Ω : these arrows are analogous to functions computing the truth tables of $\wedge$ and $\vee$ in set-theoretic case.

So it seems that in some sense, propositional logic is represented at the level of algebras of sub-objects. The point is that we do not recover structure concerning negation, in that context : lattice of sub-objects is not in general boolean. Thus every topos exhibits some propositional logic at the level of its algebras of sub-objects, but the latter does not perfectly adjust in general to classical propositional logic.

Topos semantics

New point is that we are able of mimicking Tarski's semantics in the context of toposes. Choosing an object A of a topos $\mathcal{C}$, we associate to each formula X having exactly m free variables a subobject of A^m, which stands in our minds for the subset of all m-uples satisfying formula X read 'in A'. In that purpose, we have to interpret every relational constant R with n places in the language of X as a subobject of A^n as well. It appears to be convenient, in that endeavor, to identify sub-objects through their characters.

Indeed we have to use a recursive definition of our 'validity sub-object' of X, following construction steps of X. In that definition we use logical arrows mentioned earlier, letting them act on characters of previously obtained 'validity sub-objects'.

We also need new arrows expressing quantification. We introduce arrows $\forall_A : \Omega^A \rightarrow A$ and $\exists_A : \Omega^A \rightarrow A$, mimicking set-theoretic maps $\forall_A$ and $\exists_A$ defined on Powerset $\mathcal{P}(A)$ by
$\forall_A(B) = 1 \text{ if } B = A \text{ and } 0 \text{ if not.}$
$\exists_A(B) = 1 \text{ if } B \neq \emptyset \text{ and } 0 \text{ if } B = \emptyset$

Famous result is that, with respect to such semantics, universally valid formulas (valid at any object of any topos) are theorems of intuitionist predicate logic. This seems to bring an argument favoring the latter agains classical predicate logic, making it appear as ‘more general’.

For us, important point is perhaps the simple fact that category theory, in the guise of topos theory, appears as easily welcoming logical structure of logical semantics.

We are now going to examine connections which rather derive from looking at categories in a ‘linguistic way’ : from seeing them as enfolding as it were some language or syntactic structure.

Internal language of a category

Following Lambek-Scott's treatise, I shall take category $\mathcal{C}$, in what follows, to be a topos or at least a Cartesian-closed category. We associate to such categories a 'type theory', that we understand as its 'internal language'. Objects of $\mathcal{C}$ we take as naming types of our language : thus we have composed types like $A \times B$ or B^A for given types A and B. Terms of our language we understand as arrows of the category : an arrow $a : 1 \to A$ will be a term of type A, and $x : 1 \to A$ [naming an indeterminate arrow] will be a variable of type A. For those terms category structure gives us rules of formation, like the following : if $a : 1 \to A$ and $b : 1 \to B$, then $\langle a, b \rangle : 1 \to A \times B$.

We also associate 'categorical truth values' for assertions of our language, which enjoys thus a kind of intended semantics, concerning category $\mathcal{C}$ itself. For example, if $t : 1 \to A$ and $t' : 1 \to A$ are two terms of the same type A, equality assertion $t = t'$ will be assessed by arrow $\delta_A \circ \langle t, t' \rangle$, where δ_A is the character of diagonal sub-object $\langle 1_A, 1_A \rangle : 1 \to A \times A$. Thus $\delta_A : A \times A \to \Omega$ and $\delta_A \circ \langle t, t' \rangle : 1 \to \Omega$ is indeed a truth value. Using exponentiation, we manage to find an arrow expressing truth value of assertions of the form $t \in \alpha$ when t and α have appropriate types.

It then happens that internal language of a topos allows us to formulate properties of the category itself. For example we have the result that arrows $f : A \to B$ and $g : A \to B$ are equal arrows iff

$$\forall^A x \ fx = gx$$

This seems to restore set-theoretic meaning of arrows. quantifier $\forall^A$ quantifies over objects of type A, corresponding to arrows $x : 1 \to A$ at the level of the category.

Conclusion would be that there is a (type-theoretical) language immanent to the category, seeming to reflect its categorical structure, which is at the same time able to formulate categorical facts : our language describes the categorical world it originates from.

Logico-linguistic vision of categories

We may also see categories as deductive systems. Now we regard objects of $\mathcal{C}$ as formulas, and arrows $A \xrightarrow{f} B$ as statements according to which f is a proof deriving B from A. That a composition for arrows is given appears now as a rule of inference, like follows.

$$\frac{f \ : \ A \to B \qquad g \ : \ B \to C}{gf \ : \ A \to C}$$

Which we may read as meaning that we have chained proofs f and g.

In order to make our deductive system mirror categories, we have to add axioms specifying properties of categories, or making the category, eventually, a Cartesian closed one or a topos. Such axioms take the form of equations bearing on arrows. For example, we will have the fact that $(hg)f = h(gf)$ for $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$.

Lambek and Scott make here a historical point, more or less reconstructing the story leading to Curry-Howard correspondence. They begin with properties of morphisms in a Cartesian closed category, which lead them, applying Ockham's razor they say, to a purely algebraic perspective (and to Schönfinkel and Curry). At some point the necessity of speaking in terms of types is met, which makes the junction with type theories (known since Russell and Whitehead). Everything is at the same time understood in terms of categorical notions.

Further in the chapter, they address the decision problem for arrows : do we have an algorithm for deciding whether two arrows expressed in terms of the same indeterminate $1 \xrightarrow{x} A$ are equal ?. They show that such problem may be solved with the help of Church-Rosser theorem, about reducibility of λ-terms in lambda-calculus. One has the feeling that category theory is fed with logical contents belonging to proof theory or calculability theory.

No doubt that a lot more could be said along those lines. Matters of logic have shown an incredible ability of being translated into categorical world and language. Goldblatt speaks about a “categorical analysis of logic”, but one has the feeling that frontier between category theory and logic gets blurred, that their distinction threatens to disappear.

A Categorical Turn of Philosophy of Mathematics ?

In order to deal with the issue, I shall use the list of questions describing agenda of philosophy of mathematics according to *Philosophie des mathématiques* (Vrin, 2008) .

1. How should be formulate the distinction and demarcation between philosophy and mathematics ?
2. How should we characterize the status of mathematical object ?
3. How should be formulate the distinction and demarcation between logic and mathematics ?
4. Which understanding are we able to provide for history of mathematical thought and mathematical object ?
5. How should we describe and how should we comment on 'geography' of mathematics, its division into branches ?

If their is indeed a 'categorical turn', it should manifest itself at the level of at least one of these questions. I leave for now question 1, which I will address only at the end.

Question 2

As we saw, category theory was introduced in the beginning as a new language rather than a new kind of object. Clearly, category theory worked very much in that way, allowing new (levels of) structural reflection of mathematics, motivating new generative procedures, collaborating on the whole nicely with set-theoretic presentation of mathematical objects.

Still, some people raise the foundational issue, suggesting that we could get rid of set theory, and formulate everything from the beginning at the level of category theory. It can be shown indeed that we may translate all of ZF into ETCS (theory of well pointed topos with Axiom of choice and Natural Number Object), which seems to alleviate the claim.

One can formulate objections though.

1. Category theory does not provide a synthetic picture of mathematical objectivity like set theory does with inverted cone of cumulative hierarchy of ranks (classical remark).
2. Semantics is better related to set theory than to category theory. Even in topos semantics, we have to consider interpretations and assignments as given as 'maps' : set theory may include them when it works as meta-theory for semantics, which seems more difficult for topos theory, at least directly.

But then category theory could still inspire a new philosophical vision of mathematical object. It could be argued, for example, that category theory introduced a systematic possibility of ‘thematizing’ (converting into a thought object) any kind of structural feature of mathematics, which brings a new status for the object. Indeed category theory allows such thematizing in a quite spectacular way (every categorical object seems to express structure or structure correspondence).

Still, does not set theory already do the job ? Set-theoretic constructions often reflect structural settings : the center of a group 'locates' and 'objectifies' universal commutation inside the group, for example. Kernel of a linear morphism objectifies lack of injectivity. What category theory brings at the level of status of object seems to be an improvement or a radicalisation, but not a revolution.

Another option would be to consider here Lawvere's idea of an 'internally developing set'', but I think it has more to do with question 4.

Question 3

A first observation is that despite impressive captation of logic by category theory, it seems impossible to sustain that the latter indeed swallowed the former. People are still working in the field of logic also without categories, even in areas like proof theory or calculability, where categorical approaches exist. Huge developments like those undergone by *Model Theory* in the direction indicated by Shelah's papers look like independent of categorical views or tools.

Maybe what we can say here is that we are rather witnessing typical cases of interaction between mathematics and logic, taking category theory as part of mathematics.

1. On the one hand, category theory shows logical tools working inside 'intuitive' mathematics. Such technical collaboration goes beyond foundational status of logic. With Deligne's theorem saying that a coherent topos has 'enough points', a theorem which is deeply connected with compactness and completeness theorems in logic, or with decision problem for arrows getting solved by Church-Rosser theorem, we seem to have new examples of that.
2. On the other hand, category appears as one of these mathematical theories enjoying greater abilities to welcome and translate logical issues : they fit more to what could be called the discrete and finitary skeleton of mathematics (like group theory or simplicial topology). They stand nearer from 'linguistic strata'.

In both cases, what we are describing is new interaction between mathematics and logic, that we can illustrate also outside of category theory.

But all that precedes does not change, I guess, our conception of demarcation. Even now, when mathematics and logic share the same type (formal, deductive) of exposition, we distinguish logic as more concerned with language and less with object, and as having to deal with truth as such, while mathematics has rather to explore structure as such. Philosophy of demarcation remains the same, even if, it has to be admitted, something in some sense intermediate has arisen with category theory.

Question 4

As I already outlined, Lawvere's notion of 'internally developing set', defined as a functor $P \to Set$, where P is the category associated with some ordered set, has to to with 'historical' issue. Indeed Lawvere interprets that $F(p)$ is the set as we see and know it at state of knowledge p : this sends us into the direction of a 'logic of discovery'. According to such logic, we are not supposed to deal with achieved and given objectivity, we rather unravel objects along the path of knowledge. Such conception was in part Brouwer's one, and intuitionist logic may claim to be the 'logic of discovery' we are looking for.

Grzegorczyk made that point in an important paper long ago, and Fitting, commenting on Cohen's forcing, explained the relation between intuitionist and classical validity in similar terms in a very clear little book. Proof of equivalence of Kripke-semantics for intuitionist logic with validity in topos of the form Set^P for an ordered set P connects Lawvere's ideas with such theoretical view of 'mathematical historicity' (Goldblatt, p. 223-227). Only it has to be recognized that historicity here is considered at a very abstract and theoretical level.

Quite differently, category theory may be used as another example of what I have called 'hermeneuticity' of history of mathematics.

Indeed, Mumford, using a quote from Hasse, describes in his 'little red book' how Grothendieckian concept of scheme was exactly what mathematicians were expecting and anticipating without really conceiving it : the path from Hasse to Grothendieck seems to be a path from pre-understanding to understanding.

On the whole, it does not seem that category theory imposes any brand new conception of historicity of mathematics. It rather confirms or illustrates already formulated views.

Question 5

Category theory has much to do with contemporary geography of mathematics. In its first use, as we saw, it shows some structural unity among various branches, helps to connect them with each other. It suggests as well invention of new branches, grounded on such connections (like in the cases of algebraic topology and algebraic geometry). But category theory also adds a new branch, it stands as a specific branch among other branches.

Analysis and Algebra have arisen in the modern period as 'methodological branches' rather than branches associated with some objective region. Now category theory appears as a 'branch of structural reflection', crossing all domains and levels of mathematical objects and structures.

Analysis and Algebra have become, along centuries, kind of 'meta-branches' gathering a lot of sub-branches, dividing the whole of mathematics while having their overlap. Category theory is also a 'meta-branch', but from another type : really 'meta', adding structural reflection to the whole building, while co-working at perpetual extension of constructions.

Question 1

I am now addressing question 1, which I have until now kept aside. We are asking whether category theory modifies our conception of demarcation between philosophy and mathematics. I think there is a surprising way of contending it does : we may claim it changes our way of thinking of that frontier inasmuch as it justifies Hegel's philosophy.

Indeed, category theory looks like a reflection of mathematics arising from inside and manifesting, as structural reflection, its essence. We may also describe it as connecting mathematics with its totality. Isn't it what *concept*, as ultimate substance/subject, does in Hegelian philosophy ? (It does so following the journey of its development)

In the same vein, category theory begins with the insight that their morphisms are essential to objects : apparently it enfolds a 'relational ontology' of mathematical objectivity. Again, one can argue that this is exactly how Hegel pictures and understands objects in general, as finding and asserting their essence in all kind of relations they are involved in.

The strange proximity of category theory with logic can be understood in a similar way. According to Hegel, logical form cannot be separated from logical content, both of them have to be grasped at the level of (conceptual) development, which is the universal way of self-presenting ontology. Thus logic requires a 'narrative' of its categories, exposing them as moments of the process of being.

Category theory, when capturing and translating every possible logical notion, would prove that these notions were ontological from the beginning (and would allow their dynamical reading). Category theory would disproof the disjunction between logic and ontology : a disjunction we could claim on the contrary *Model Theory* was re-asserting.

This would be a formidable lesson about demarcation issue. It would teach that with category theory, mathematics has become philosophy in the best possible way, achieving Hegel's program. It has to be reminded here that Hegel was contrasting mathematics and philosophy very strongly. He saw mathematics as 'deprived of thought', because it was unable *not to* treat everything as 'intuitive' and separated.

Instead of marveling at such grandiose encounter, some arguments may help us resist it, though.

- Nothing, in category theory, reminds us of ‘logic of contradiction’ : category theory works by reflection rather than contradiction, which makes a huge difference.
- Category theory is not only and radically *movement*. In some sense, it asks us to forget about the dynamical meaning of maps (transformation of x into $f(x)$) by ‘flattening them’ into arrows.
- Even if category theory may bridge every part of mathematics with each other, it does not forget of ‘regional’ specificity, which it may highlight in some cases. It does not seem to blindly obey the logic of totality.

As a comment to that last point, I shall go back to that staggering exercise of MacLane's treatise I exposed earlier. Maybe what I felt was a Hegelian vertigo. I had the feeling that category theory, by mirroring everything with everything, was destroying rationality as I like it, in the name of Hegelian totality and universal 'relatedness'. Still, even in that example, we do not lose after all the clear distinctive conceptions of algebraic and order structure (in agreement with what I said in the last item of preceding slide).

Thus conclusion will be that there is no 'categorical turn' concerning question 1 either. Still, maybe, category theory reminds us that, as much as philosophy separates and distinguishes itself of mathematics, it remains continuous with it at the same time.