On the information contained in representations

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My question: To what extent can B&E’s informational view of representations account for mathematical practice, including discovery?

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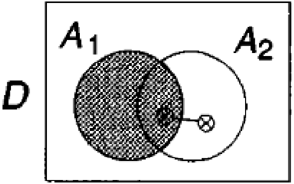
The sets used for these two purposes may not coincide.

Information content, first use: defining models

Example: Venn diagrams in Shin (1994)



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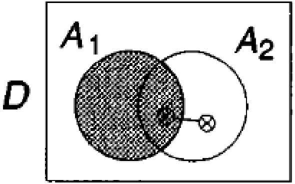

Representing facts of D:	A set assignment satisfies D if the corr. represented facts hold:
Region A_1 is shaded	The set corr. to A_1 is empty
Region A_2 has Xs	The set corr. to A_2 is nonempty

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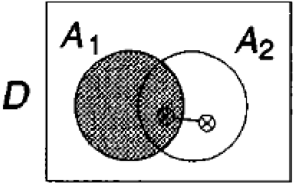

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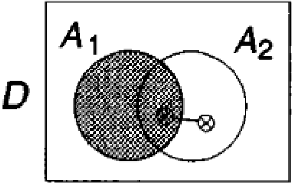

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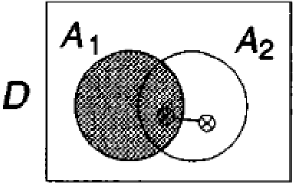

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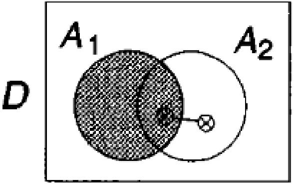

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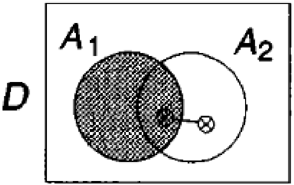

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(In this example, basically the same "content" is used to define the semantics and to set up Observe rules.)

II. The problem of patterns

A problem: Typically, we see more in diagrams than the information they are supposed to contain according to these systems.

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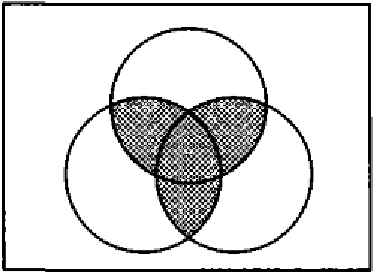

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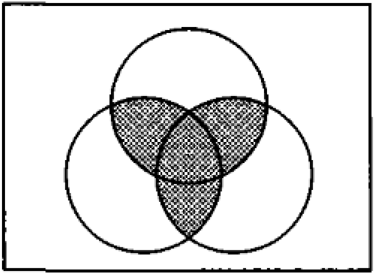

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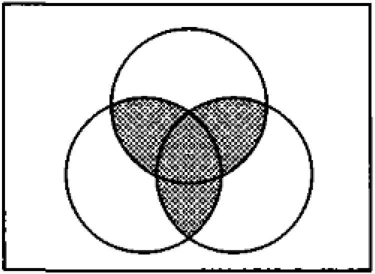

The sets are pairwise disjoint. They play symmetric roles.

“Conjunctive” patterns (2)

This is the group table of Klein’s four-group:

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

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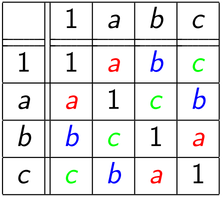

	1	*a*	*b*	*c*
1	1	*a*	*b*	*c*
a	*a*	1	*c*	*b*
b	*b*	*c*	1	*a*
c	*c*	*b*	*a*	1

The group is commutative.

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Ulam’s diagonals

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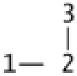

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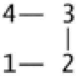

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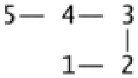

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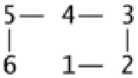

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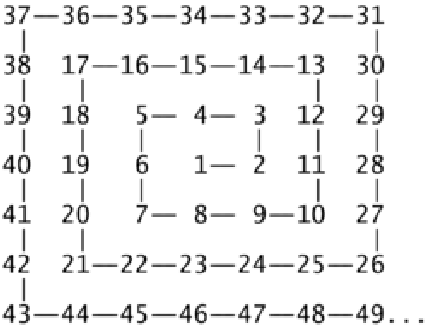

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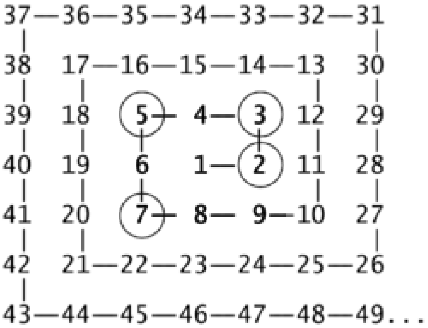

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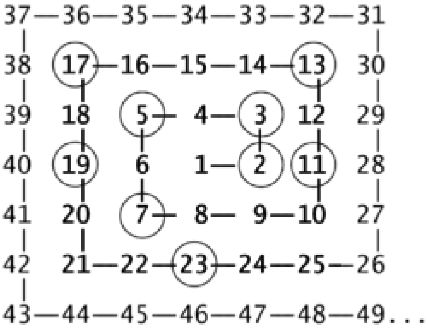

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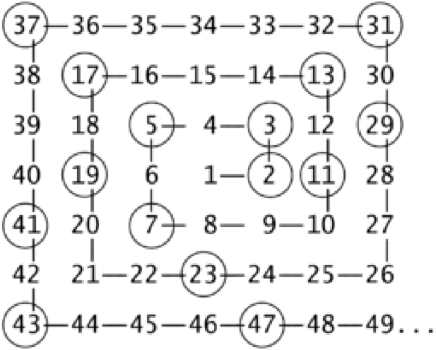

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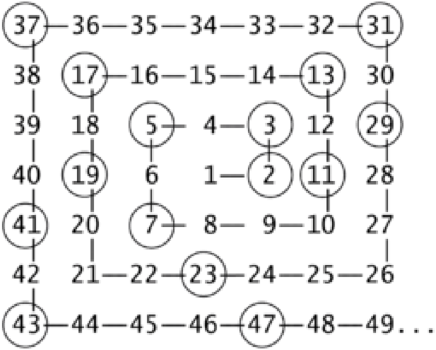

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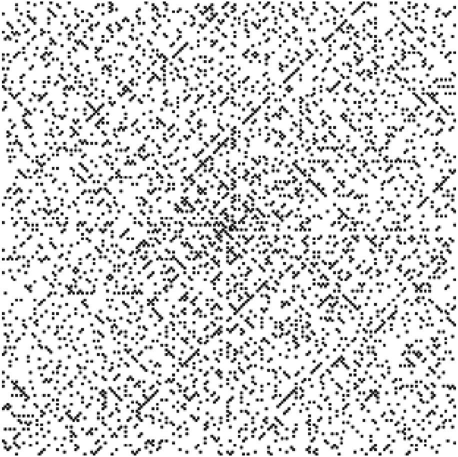

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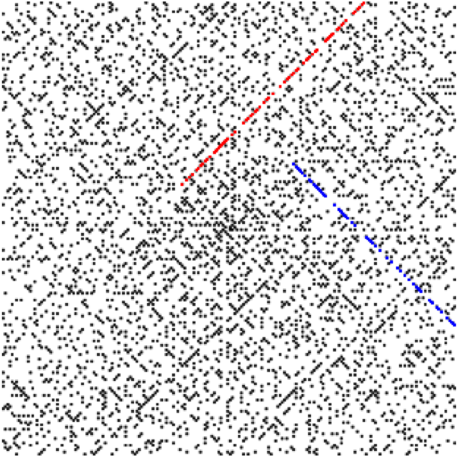

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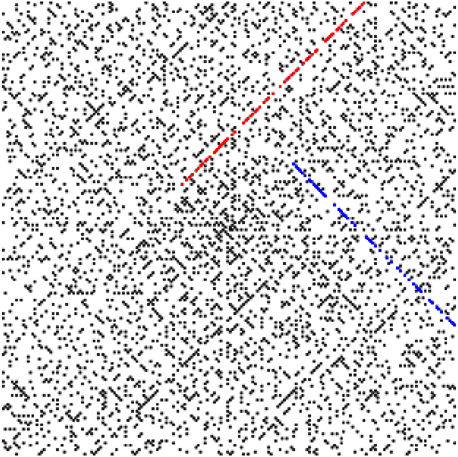

These diagonals correspond to values of $4n^2 + bn + c$ for some b, c. It appears that some of these sets contain an unusual concentration of primes.

More complex patterns (1)

There are also patterns that are *prima facie* not reducible to a conjunction of more basic information. One can think, for instance, of symmetries in geometrical diagrams: it seems difficult to me to reduce these symmetries to the sentences that replace the diagrams in Avigad, Dean, and Mumma (2009).

More complex patterns (2)

Rolle's lemma and the mean value theorem

Let $f : [a, b] \to \mathbf{R}$ be a differentiable function.
Rolle's lemma: if $f(a) = f(b)$,
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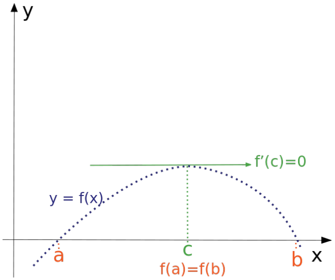

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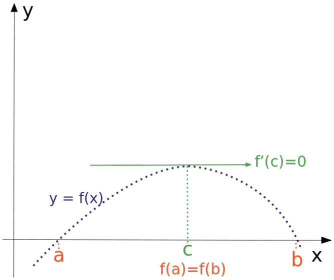

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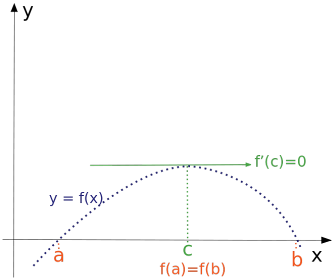

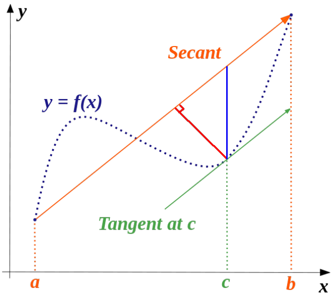

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- We look at polynomials to determine their degree.
- $d = xy + yz + xz$. We can notice that there is a permutation symmetry between x, y and z.

Patterns in formulas (2)

Leibniz and the analogy of powers and differences

A real-life example from the historical case study I originally meant to present: Leibniz notices a vague analogy between

$$\int \overline{z^e d^m n} = z^{e} d^{m-1} n - e z^{e-1} d^{m-2} n dz + e(e-1) z^{e-2} d^{m-3} n \overline{dz}^{2} - e(e-1)(e-2) z^{e-3} d^{m-4} n \overline{dz}^{3} \text{ etc.}$$

and

$$(A+B)^z = A^{z} + \frac{z}{1} A^{z-1} B^{1} + \frac{z(z-1)}{1.2} A^{z-2} B^{2} + \frac{z(z-1)(z-2)}{1.2.3} A^{z-3} B^{3} \text{ etc.}$$

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Moreover: These examples also show how we engage in meta-representational reasoning: when we try to understand what a given pattern might mean, we reason about the link between the representation and what we take it to be about.

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In contexts of mathematical discovery, this is not always the case. We may need to experiment with formulas or diagrams with no clear semantics. We may only be able *a posteriori* to provide such a semantics – and perhaps only a partial one.

Non-representational uses of formulas and diagrams

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- $d^0y = y$ and $\frac{d^2y}{d^2y} = d^0y = y$;

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- $d^0y = y$ and $\frac{d^2y}{d^2y} = d^0y = y$;
- Also $\frac{y}{y} = \frac{d^0y}{d^0y} = d^0y = y$;

Non-representational uses of formulas and diagrams

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Bernoulli astutely uses these rules to compute integrals, and gets correct results. In fact, his methods are only valid in very limited cases, and accounting for that requires a fair amount of reformulation and reinterpretation.

It is only *a posteriori* that we can see these formulas as representations carrying definite pieces of information.

Recapitulation

Introduction: Barwise and Etchemendy's program

I. A clarification: What is the information "contained in" a representation?

II. The problem of patterns

- "Conjunctive" patterns
- More complex patterns
- Patterns in formulas
- Lessons

III. The problem of non-representational uses of diagrams and formulas

- Johann Bernoulli's strange computations