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3- Description du projet de recherche – Description of the research project

Fields of research project : Logic, History and philosophy of logic and mathematics

Description of the research project and its scientific goals

Summary. The proposal consists in research and activities related to the history and the philosophy of mathematics in a set of areas connected to infinity. The general context of my approach is the philosophy of mathematical practice, namely an approach to the philosophy of logic and mathematics that pays great attention to mathematical practice (in any area of mathematics) as a source of philosophical reflection. Attention to mathematical practice also involves attention to the historical roots of the problems investigated, as the historical record just is the record of our mathematical accomplishments. Starting from a recent development in mathematics, the theory of numerosities, I propose a broad range of historical, mathematical and philosophical investigations that are deeply interconnected. The motivation for such investigations is twofold: on the one hand to probe the sources of our knowledge of numbers and probabilities and, on the other hand, to make novel and far-reaching contributions to some of the most central areas of discussion in contemporary philosophy of mathematics and philosophy of probability as they relate to infinity.

Description

Background. The proposed project is a natural extension of a set of concerns that have been a kind of invariant in my scholarly activity: attention to mathematical practice and a long-standing concern with infinity. But before I enter the details of the proposal, I would like to emphasize the following. In addition to the methodological 'style' described above (the philosophy of mathematical practice) and the overarching engagement with infinity, there is a deep unity that ties the investigations constituting this proposal. I am after a deeper understanding of counting and reasoning with probabilities, two of our most basic cognitive capacities.

The research project.

Mancosu 2017 (see the longer description for the bibliographic references) explored how the recent theory of numerosities (Benci and Di Nasso 2003) allows one to extend counting from finite to infinite sets in such a way as to preserve –in contrast to Cantor's notion of cardinality– the part-whole principle (as it had been propounded, for instance, by Bolzano): *If a set A is strictly included in B, the 'size' or 'numerosity' of A should be strictly less than that of B.* In Cantor's assignment of cardinalities to infinite sets this principle does not hold since the natural numbers and the even numbers (which are contained properly in the former set) have the same cardinality. This mathematical development led me to a set of new areas of investigation that I describe in three separate sections although the topics are intertwined.

Section I. The History of Infinity. Counting at infinity: part-whole, one-to-one correspondence, and frequency.

As I have shown in the work culminating in my 2017 book, the history of infinity (and the notion of sameness of "size" for collections/sets) has been hostage to the complete dominance of Cantor's notion of cardinality and the associated criterion of sameness of "size": two sets have the same cardinality if and only if there is a one to one correspondence between the elements of the sets. Let us call this criterion ONE-ONE. An alternative criterion, defended among others by Bolzano, rests on the part-whole intuition: if a collection A is strictly included in B then the "size" of A should be strictly less than the "size" of B. Let us call this criterion PART-WHOLE. It is obvious that while agreeing on finite collections, ONE-ONE and PART-WHOLE disagree when it comes to determining the "sizes" of infinite collections. The natural numbers and their squares have the same size (cardinality) according to ONE-ONE but different sizes according to PART-WHOLE. And this is the essence of Galileo's paradox for numbering infinite collections described in *Two New Sciences* (1638; 1974; see also Knobloch 1999). Despite Bolzano's attempts, PART-WHOLE was not implemented as a coherent mathematical theory until the work on the theory of numerosities by Benci, Forti and Di Nasso, in the early 2000s. Cantorian set theory, and its concomitant definition of sameness of size based on ONE-ONE, was so successful – indeed, it was the only successfully worked out theory of "size" for infinite sets – that a sort of consensus developed, to which Gödel gave a prominent voice, according to which any generalization of counting from the finite to the infinite had inevitably to yield the Cantorian solution. I have debunked Gödel's inevitability argument in my 2017 book but in this part of the project I am concerned with emphasizing the devastating influence that relying on Cantorian set theory has had on the history of infinity. Indeed, the same kind of Gödelian conviction about the inevitability (and thus ultimate "correctness") of Cantor's solution has been influential in the kind of "Whig" history of the infinite that many historians have perpetrated. What I mean to point out are two symmetric mistakes that have gravely affected many accounts of the history of infinity, namely either to assign high praise to a past thinker on account of his conception of infinity having foreshadowed Cantor's construction or, alternatively, condemnation for his not having done so. I will not repeat here the extensive evidence that I provided in my previous work (2009, 2017) when tracing the genealogy of ONE-ONE and PART-WHOLE in the history of infinity . But it was during that research that I noticed a third way of counting at infinity that has so far remained at the margins of our historical consciousness and whose centrality has once again remained unperceived on account of the widespread Cantorian viewpoint I have described. I would like to call this principle FREQUENCY.

The first occurrence of FREQUENCY I am aware of is in the work of the Islamic mathematician Ibn Qurra (IXth century A.D.). Ibn Qurra states that odd numbers and even numbers have the same size. But one should be careful not to immediately read his argument as being the standard Cantorian one based on one to one correspondence, for the motivation adduced does not generalize to other arbitrary infinite sets. Rather, it would seem that some informal notion of frequency (how often do even numbers, respectively odd numbers, show up?) is in the background of ibn Qurra's conception of infinite sizes ("we find in every three consecutive numbers one that is divisible by three"). My proposed project in this area consists in retracing the frequentist intuitions in the history of infinity. Starting with ibn Qurra, I will then move on to Robert Grosseteste who in his treatise *De Luce* (approximately 1220) proposes an arithmetic of numerical infinities that has remarkable similarities with that proposed by ibn Qurra. I would like to retrace the history of the frequentist intuition passing through Galileo, Maignan, Bolzano and the development of the notion of density in number theory (thereby providing novel information on the emergence of the notion of asymptotic density in number theory). The upshot will not only be the new historical information that such a study will be able to provide but, just as importantly, one important dividend will be that of bringing to full philosophical awareness that in addition to ONE-ONE and PART-WHOLE the history of infinity needs to take into account the role of FREQUENCY. This is obviously related to my general concerns about our sources of the notion of counting at infinity and its connection to probability theory.

Section II. Philosophical Investigations

II.A Abstraction and Infinity: assigning numbers to infinite concepts.

As already mentioned, in Mancosu 2009, 2015a, b, 2017, I have explored the historical, mathematical, and philosophical issues related to the new theory of numerosities. In 2015a, b, 2017, I generalized some specific worries emerging from the theory of numerosities to a line of thought resulting in what I called a 'good company' objection to Hume's Principle (which assigns 'sizes' to concepts – this is the Fregean equivalent of subset of the domain – using the Cantorian criterion of one to one correspondence). Let me begin by recalling a few central facts about Hume's Principle and neo-logicism. Neo-logicism (Hale and Wright 2001) is an attempt to revive Frege's logicist program by claiming that important parts of mathematics, such as second-order arithmetic, can be shown to be analytic (or akin to analytic). The claim rests on a logico-mathematical theorem and a cluster of philosophical arguments. The theorem is called *Frege's theorem*, namely that second-order logic with a single additional axiom, known as Hume's Principle, deductively implies (modulo some appropriate definitions) the ordinary axioms for second-order arithmetic. The cluster of philosophical claims are related to the status (logical and epistemic) of Hume's Principle. Let us recall that in a Fregean context the second order systems have variables for concepts and objects (individuals). In addition, we have functional symbols that when applied to concepts yield objects as values. One such functional symbol is #. The intuitive meaning of # is as an operator that when applied to concepts outputs objects corresponding to the cardinal number of the objects falling under the concept. For instance, '#x:(x=Barack Obama)' denotes an object that will be taken to be the number 1, where 'x=Barack Obama' expresses the concept of being identical to Barack Obama. Hume's Principle (henceforth HP) has the following form:

HP $(\forall B)(\forall C)$ [$\#x:(Bx) = \#x:(Cx)$ iff $B \approx C$]
where $B \approx C$ is short-hand for one of the many equivalent formulas of pure second order logic expressing that "there is a one to one correlation between the objects falling under B and those falling under C". Informally, it can be read as saying that two concepts B and C have the same 'number' if and only if there is a one to one correspondence among the objects that fall under B and those that fall under C. Principles like HP that define a function from an equivalence relation are called abstraction principles. Another abstraction principle that will be needed later in our discussion is Basic Law V, an inconsistent abstraction (as Russell showed), postulated by Frege in his *Basic Laws of Arithmetic*. It has the following form:
BLV $(\forall B)(\forall C)$ [$\varepsilon x:(Bx) = \varepsilon x:(Cx)$ iff $\forall x(Bx \leftrightarrow Cx)$]
In short, two concepts have the same 'extension' ($\varepsilon x:(Bx)$ stands for the extension of the concept $B(x)$) if and only if every object that falls under the first falls under the second and vice versa.

It is important to point out that both HP and BLV play the role of axioms of infinity. When adding HP to second-order logic we automatically generate all the natural numbers. The same happens with BLV, which yields infinitely many objects, although on account of its inconsistency (about which I will say more later) this is not as impressive a result as the one about HP.

In 2015a,b, 2017 I showed that there are a countable infinity of abstraction principles that are 'good', in the sense that they share the same virtues of Hume's Principle (they are consistent etc.) and from which we can derive the axioms of second order arithmetic. This is an important technical result as well as a philosophical one, for hitherto only Hume's Principle was thought capable of such a feat. I then articulated a 'good company' objection to the effect that no such principle can be analytic, for they are all consistent but they knock each other out. Finally, in my investigations, I provided a tentative taxonomy of possible neo-logicist responses to the 'good company' objection and claimed that among the three positions I singled out (conservative, moderate, and radical neologicism), the moderate position was the one with the best chance of coming out unscathed from the 'good company' objection. This was partly based on my evaluation that the moderate position could address the problem of cross-sortal identification for abstracta yielded by different abstraction principles (see below) better than the conservative and the radical position. I was led to this position by the fact that all the good companions of Hume's Principle that I had individuated agreed in the definition of the finite numbers (i.e. all the principles relied on one-to-one correspondence for determining when two finite numbers are identical). However, recent discussions with specialists in this area have led me to refine my 'good company' objection in such a way that even the moderate neo-logicist is saddled with a major problem. While I cannot rehearse the details let me point out that the idea is to develop the natural numbers using an abstraction principle called "Finite Basic Law V", a consistent modification of Frege's Basic Law V obtained by restricting the range of the concept quantifiers in Basic Law V to finite concepts. Let me emphasize that Finite Basic Law V, unlike Basic Law V, is consistent. This move is similar to the use of "Finite Hume's Principle" for developing the natural numbers, which had been established by Richard Heck in Heck 1997 (again, this is obtained by restricting the range of the concept quantifiers in Hume's Principle to finite concepts.). Then the ontological issue for the moderate neo-logicist becomes that of articulating on what basis one can identify the natural numbers yielded by the first principle (Finite Basic Law V) with the natural numbers yielded by the second principle (Finite Hume's Principle). For instance, on what basis can we identify the 0 obtained as $\#x(x \neq x)$ (using Finite Hume's Principle) with the 0 obtained as $\varepsilon x:(x \neq x)$ (using Finite Basic Law V)? This is the problem of cross-sortal identification of abstracta (see Cook and Ebert 2005). In order to articulate the objection precisely one will need to show in detail how to generate the natural numbers from "Finite Basic Law V" (as I mentioned, the equivalent result for "Finite Hume's Principle" is due to Heck and dates from 1997) and then spell out how the metaphysical challenge for cross-sortal identification of the two sorts of natural numbers is supposed to work. The generation of the natural numbers from Finite Basic Law V can be obtained constructing the natural numbers as Zermelo ordinals, i.e. using concepts under which only one element falls. I have a very clear idea of how I want to proceed but some of the results need to be thought out more carefully and written in publishable form.

II B. Counting and Probability

The existence of alternatives to Hume's Principle (i.e. the good companions of Hume's Principle) mentioned in part IIA of the project leads to a more general set of worries.

1) If we have different conceptual possibilities concerning the generalization of counting from finite to infinite numbers, what criteria become relevant for favoring one account over the other?

2) How does the new theory of numerosities help give us insight (or even solve) the problem of 'God's lottery' (see below) in probability theory?

II B.1 The problem of 'fruitfulness' and the conceptual nature of counting.

This set of issues takes its start from the kinds of considerations adduced in the literature for claiming the superiority of Cantorian notions of 'size' (cardinality) for infinite sets vs. other notions, such as those defended by Bolzano (which can be partially vindicated in the theory of numerosities). For instance, if – contra Gödel –we abandon the thesis of the inevitability of the Cantorian generalization to infinite numbers, then other criteria must play a role in favoring one theory over another. If 'fruitfulness' is the criterion for choosing between different accounts, one should spell out what this concept of fruitfulness amounts to, if not in general at least in the specific case. There are, for instance, interesting applications of the theory of numerosities to density in number theory (see Di Nasso 2010) and probability theory (see below) that might serve as a starting point. Of course, the discussion would have to be conducted in constant comparison with arguments for the mathematical fruitfulness of Cantorian notions of size in various mathematical domains. In addition, one can make the picture more subtle by taking into consideration the FREQUENCY principle.

A related investigation has to do with whether theories of numerosities can be good theories of counting and how they fare in comparison to Cantorian theories (see also Parker 2013). For instance, it is obvious that numerosities are not invariant under translations (a mapping sending every natural number n to n+1 will not preserve the 'size' (in the sense of numerosity) of the domain and the range of the function). In addition, counting with numerosities seems to present much arbitrariness due to the nature of the mathematical construction required (which rests on special kinds of non-constructive ultrafilters). This investigation is more radical than the former one, for it addresses not only issues of mathematical fruitfulness (for instance, whether one of the two theories is more useful in real analysis, number theory, or topology) but deeper conceptual issues about the nature of counting. My goal here is to pursue a systematic exploration of both topics. I have also been delving into the literature on cognitive psychology to investigate whether 'counting by groups' (an intuition that is behind the theory of numerosities) or using frequency intuitions might afford good empirical evidence to look at these problems.

Related to the issue of fruitfulness is also the possibility of conceptual gains in probability theory, which I describe next.

II B.2 God's lottery, probabilities and numerosities.

In an article titled 'God's lottery', Storrs McCall and D.M. Armstrong (1989) considered a lottery in which God chooses a number from the set of all positive integers (this is also called a de Finetti's lottery). God's lottery is assumed to be fair in that the chance of any given number being chosen is the same as that of any other number. Within Kolmogorov's axiomatization of probability there is simply no way to account for a fair infinite lottery on the natural numbers. Whenever we face a challenge of this type a delicate foundational and conceptual balancing must take place. Obviously one or more of the principles holding for fair finite lotteries must be given up. But which principles we give up can only emerge from a subtle dialectic between mathematical coherence and intuitive desiderata. In this part of the project I investigate new theories of infinitesimal probability (which emerged from the theory of numerosities) and explore how one should weigh different desiderata in generalizing our intuitions from finite to infinite lotteries. Among the things I intend to investigate are non-denumerable lotteries and the issue of how different implementations of the theory of numerosities might satisfy important intuitive desiderata on God's lottery. There is plenty of conceptual work to be done in this area and I propose to probe such issues further. Among other things, I am interested in analyzing whether different conceptions of probability, subjective vs. objective, are differently affected by these recent developments.

SECTION III. Mathematical investigations.

The investigations to be described in this section find their roots in my studies in mathematical logic. The key theme I am after here is the surprising role that infinitary considerations play in establishing results about the finite. This is a topic that also intersects with the philosophy of mathematical practice, for the role of appealing to infinitary principles for establishing results about the finite is key in discussions of purity (why should infinity enter into the proof of statements that are apparently only about finite entities?) and explanation (does infinity play an explanatory role in proving results about the finite?). The issue first emerges in connection to Peano Arithmetic.

Gödel's first incompleteness theorem shows that under the assumption that PA is consistent (and minimally sound) one can find a statement such that neither it nor its negation is provable in PA. While perfectly fine for the logician's needs, the Gödel sentence, call it G, appears concocted from the point of view of the practicing mathematician. Since the mid-seventies, logicians have been able to find statements with obvious mathematical significance that are independent of PA (results by Paris-Harrington on modifications of finite Ramsey's theorem and Paris-Kirby on Goodstein's sequences). Establishing the truth of the Gödel sentence and of the new incompleteness results requires appeal to some "infinitary" principles (when the truth of the Gödel sentence G is established through appeal to the statement expressing the consistency of PA, it is establishing the latter that requires some portion of infinitary reasoning, such as induction up to an ordinal called epsilon-zero). Recursion theorists (see Hirschfeldt 2015) are also trying to understand the role of infinitary principles or compactness arguments (such as those relying on Weak König's Lemma) play in our determination of results about the finite. I would like to extend the investigations contained in Mancosu-Siskind 2019 and Siskind-Mancosu-Shapiro 2020 on abstraction principles and choice principles to study what more we get, in terms of additional arithmetical knowledge, by using abstraction principles that provide conditions of identity for abstracta obtained from finite and infinite concepts in comparison to the corresponding restricted abstraction principles that only specify identity conditions for abstracta obtained from finite concepts. An analogue and well understood question is that concerning what "extra" arithmetical content the system of ZF (which has an axiom of infinity) provides in comparison to the system ZF-Inf (i.e. the system of Zermelo Fraenkel set theory where the axiom of infinity is removed; the latter system is equivalent to PA). The first examples to study will be the systems characterized by the abstraction principles described above as Finite Hume's Principle and Hume's Principle. Can we cash out the additional content that Hume's Principle has with respect to Finite Hume's Principle in precise arithmetical terms? I can see how the tools used in Mancosu-Siskind 2019 and Siskind-Mancosu-Shapiro 2020 (which consider abstraction principles as choice principles) can be enlisted for these goals but the logical research can be pushed much further as the problem I posed is, unlike that for ZF and ZF-Inf, hitherto unexplored. Philosophically, results in this area would deliver much insight concerning the philosophical discussions that occupied Boolos, Wright and Heck in the 1990s on the "excess content" of Hume's Principle with respect to ordinary arithmetical intuition.

Summary of scientific goals

To make novel and far-reaching contributions to some of the most central areas of discussion in contemporary philosophy of mathematics and philosophy of probability as they relate to infinity. The projected results results include decisive contributions to, among other things, the history of our conceptions of infinity, neo-logicism in philosophy of mathematics, the study of abstraction principles as infinitary principles yielding finitistic results, alternative conceptions of probability, the nature of concept extension, and the sources of our knowledge of counting and probabilistic reasoning. In addition, I am probing the "mysterious effectiveness" of exploiting the infinite in establishing results about the finite. My method is that of conceptual philosophical analysis coupled with historical, mathematical, and empirical investigations relevant to the matter at hand.

I intend to publish four long articles, or perhaps a book, on the results of the investigation.