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Abstract

This paper argues in favor of a nonreductionist and nonlocal approach to the philosophy of mathematics. Understanding of mathematics can be achieved neither by studying each of its parts separately, nor by trying to reduce them to a unique common ground which would flatten their own specificities. Different parts are inextricably interwined, as emerges in particular from the practice of working mathematicians. The paper has two topics. The first one concerns the conundrum of the unity of mathematics. We present six concepts of unity. The second topic focuses on the question of reflexivity in mathematics. The thesis we want to defend is that an essential motor of the unity of the mathematical body is this notion of reflexivity we are promoting. We propose four kinds of reflexivity. Our last argument deals with the unity of both of the above topics, unity and reflexivity. We try to show that the concept of topos is a very powerful expression of reflexivity, and therefore of unity.

Chapter 12
For a Continued Revival of the Philosophy of Mathematics

Jean-Jacques Szczeciniarz

Envoi. This essay is a friendly and grateful tribute to Roshdi Rashed. Needless to say, this article will deal with the philosophy of mathematics and particularly what we call recent mathematics. As a historian of mathematics, Roshdi Rashed (like the great Neugebauer) is a tireless reader of contemporary mathematics. He knows how to draw, for example from category theory examples and ways of thinking that serve as benchmarks for exploring the conceptual history of mathematics.

Abstract This paper argues in favor of a nonreductionist and nonlocal approach to the philosophy of mathematics. Understanding of mathematics can be achieved neither by studying each of its parts separately, nor by trying to reduce them to a unique common ground which would flatten their own specificities. Different parts are inextricably interwined, as emerges in particular from the practice of working mathematicians. The paper has two topics. The first one concerns the conundrum of the unity of mathematics. We present six concepts of unity. The second topic focuses on the question of reflexivity in mathematics. The thesis we want to defend is that an essential motor of the unity of the mathematical body is this notion of reflexivity we are promoting. We propose four kinds of reflexivity. Our last argument deals with the unity of both of the above topics, unity and reflexivity. We try to show that the concept of topos is a very powerful expression of reflexivity, and therefore of unity. AQ1

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12.1 Introduction

12.1.1 *Mathematics*

In his paper *A view of Mathematics* Alain Connes comments on the role of mathematics:

> Mathematics is the backbone of modern science and a remarkably efficient source of new concepts and tools to understand the "reality" in which we participate. It plays a basic role in the great new theories of physics of the XXth century such as General Relativity and Quantum Mechanics. The nature and inner workings of this mental activity are often misunderstood or simply ignored, even among scientists from other disciplines. They usually only make use of rudimentary mathematical tools that were already known in the XIXth century and miss completely the strength and depth of the constant evolution of our mathematical concepts.[1]

This is even more true of philosophy. Of course there are some exceptions like Cavaillès, Lautman, and some historians of mathematics, nevertheless one can say that the living heart of the activity of mathematics in action is generally ignored.

Our aim here is to provide some elements to change this situation. We can only propose a modest contribution in the face of the immense task that should be undertaken. The main point is to insist on the fact that an essential reason for this situation lies in the neglect or ignorance of the unity of mathematics. This unity accounts for the remarkable efficiency of new concepts and their ability to understand the reality in which we participate.

12.1.2 *Some Essential Feature of the Mathematical Landscape*

At first glance the mathematical landscape seems immense and diverse: it appears to be a union of separate parts such as geometry, algebra, analysis, number theory etc. Some parts are dominated by (various aspects of) our understanding of the concept of "space", others by the art of manipulating "symbols", and others by the problems occurring in our thinking about "infinity" and "the continuum".

This first view is not completely false, but this breaking down of mathematics into different regions of inquiry also misses much—it has a superficial aspect and needs to be rectified and re-elaborated by bringing together different elements; to go through the surface and the depth of this landscape amounts to understanding its unity. And to understand the unity is also to understand the reasons for it.

[1] Alian Connes, 2008 *A View of Mathematics*: Concepts and Foundations vol. 1 www.colss.net/ or Eolss. http://www.eolss.net/sample-chapters/c02/E6-01-01-00.pdf.

12.2 The Unity of Mathematics

The most essential feature of the mathematical world is the following: it is virtually impossible to isolate any of the above parts from the others without depriving them of their essence according to Alain Connes in the same article (Connes 2008). In order to describe this profound unity of mathematics we must take into account the nature of mathematical abstraction, and the manner in which it is related to the unity of mathematics.

According to this view the essence of mathematics is linked to its unity. The first way to think of this unity is to compare the mathematical body with a biological entity: it can function and flourish only as a whole and would perish if separated into disjoint pieces. There are many ways to think of this organic metaphor for the unity of mathematics. I would like to emphasize four aspects.

12.2.1 Four Features of Unity of Mathematics

Firstly: this first unity comes from a very old view in the history of science. It is a conception whose scope is universal and which serves to order our understanding not only of mathematics but also of the manner in which the whole physical or biological world is to be thought of as a unity. Plato, for instance, builds an organic unity, hierarchical set of entities that form the universe. The Forms (Ideas) that preside over this hierarchical unity—from the intelligible world to the sensible world—are geometric. This geometry is translated into a unit having two faces. The one is that of the universe (theory of proportions for the cosmology or for the politics, analogy of the Line[2] the other is that of the geometry itself which takes advantages for its own unity, of the intelligible world.

Penrose is fascinated by the crucial role that complex numbers play, both in quantization and in the geometry of spinors. He has always been motivated by the idea that complex structures provide an important link between these two objects. The physical universe can be explored by means of complex numbers. Moreover, complex geometry contributes to understanding the unity of mathematics. This first unity would be the unity of mathematics as the intellectual unity of science and at the same time a deepening of unification of the mathematics.

Secondly: as an organic unity, it develops from inside, just like a living being. I will say more about this feature below. It is the unity of mathematics as unity of his extension and expansion movement.

Third: this growth can take a variety of directions which carry simultaneous and multiple meanings and exhibit, so to speak, different rhythms of development. Grothendieck becomes the contemporary of Galois, Riemann of Archimedes.

[2]Plato, Rep. VI, 508–509, Platon, *Œuvres complètes* Texte établi par Auguste Diès, Paris, Les Belles Lettres, Budé T. 7-1, *Platonis Opera* John Burnet, Oxford Classical Texts, Clarendon Press 1900–1907, G. Leroux, Garnier Flammarion, Paris 2002, nelle édition de la *République* 2003.

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Fourth: the "topologies" of these different kinds of increase can take very different forms, including—for instance, metaphorically speaking—non-classical topological spaces of representation. We will see that layers of mathematics can be cut several times, (topology and algebra, integers and real numbers) or that different domains are non-separable (group theory, topological group theory, (topological) vector spaces).

12.2.2 Internal Endogenic Growth for the Second Kind of Unity

An essential feature of this organic development of mathematics is the extension of the body by new elements emerging from within, as in a living being. A theory may typically provide the resources for the expression of a further theory which develops by means of a *reflection* on the first. I will add some elements of analysis to develop this topic below. It will be our second topic.

Consider for example the calculus that Newton and Leibniz in different ways invented. It is only when it became a question for the mathematical body, when it adapted itself to a host structure produced by the body, that one arrives at "the" calculus. The production of a purely mathematical concept is the result of the absorption of a notion, arising from physical reflection, by the mathematical body. Consider the case of Leibniz's contribution, which provided a clear set of rules for working with infinitesimal quantities, allowing the computation of second and higher derivatives, and providing the product rule and chain rule, in their differential and integral forms. Unlike Newton, Leibniz paid a lot of attention to the formalism, often spending days determining appropriate symbols for concepts. And this is purely mathematical working in the sense of an internal development.

This slow process of transformation of Euclidean concepts of motion has been studied by Panza (2005).[3]

Any apparently external element, object, idea, image must be integrated and reconstructed in a mathematical manner and form. It is not certain that the calculus could have appeared without the intervention of physics, but the physical question had to be entirely transformed, mathematized, conceived as a mathematical problem in the passage from its initial Euclidean setting to the analytic one. This is the case for the concept of force and acceleration. What is important is the mathematical development of conceptual tools, whose different steps we can describe as an internalizing of external elements.

Moreover there are some difficulties with the organic metaphor. It misses a central aspect that characterizes mathematics: the fact that different disciplines have appeared that are essential for all existing mathematics. For example, topology, or algebraic geometry.... Topology impacts on all mathematics and has helped to renew old theories and approach them in a new light. Each discipline has effects on others in various ways. Thus a supplementary body appears to be essential for another. It is

[3] Marco Panza, *Newton et les origines de l'analyse: 1664–1666,* Blanchard, Paris, 2005.

possible to follow the various ways in which a discipline (such as algebra, topology, geometry. . .) leaves the marks of its growth within the body. They are able to go through very different stages of growth and roles.

12.2.3 Difficulties with the Organicist Concept of Unity, Particularly for the Third Conception: Two Opposite Concepts of the Mathematical Body

The Coming-to-be of mathematics appears as autonomous, unpredictable, and endogenous, and in accordance with a temporality such that the overall structure is out of reach. It typically involves bifurcations, branches, breaks, continuity, recovery, neighborhood relations, and moments of partial unification. We can try to propose two "optimal" forms of such development.

(a) Labyrinthine

There are many underground networks: Archimedes is related to Lebesgue and Riemann, but Archimedes is also related to Pascal and Leibniz, Lagrange to Galois and Galois to Grothendieck. There are profound underground paths, sometimes surprising. It also happens that new proofs of the same theorem come as secondary benefit of a new theory. Reintroduction of the Pythagorean theorem in infinitesimal geometry renewed its sense. Multiple timeframes are sometimes involved in this. Surprise and multiplicity of different temporalities disturb the coherence of the organic metaphor. We can nevertheless retain the affirmation of endogenous growth.

These necessarily succinct remarks on this dispersed diachronic of mathematics go hand in hand with the synchronic dimension of the mathematical body.

(b) Architectonic

There is an underground network of connections between various trajectories, whose reality or forms we do not appreciate. When these connections appear frequently and unexpectedly we can reconstruct a new region of the already known territory. We then join the architectonic organization of the mathematical body.

> Where things get really interesting is when unexpected bridges emerge between parts of the mathematical world that were previously believed to be very far removed from each other in the natrural mental picture that a generation had elaborated. At that point one gets the feeling that a sudden wind has blown out the fog that was hiding parts of a beautiful landscape.[4]

I recall some of the principal new ideas Grothendieck considered as essential to his work [R and S 1985][5]

[4] Alain Connes, A view of mathematics. *ibid.*

[5] Alexandre Grothendieck, *Reaping and Sowing* 1985 Récoltes et Semailles Part 1. The life of a mathematician. Reflections and Bearing Witness. Alexander Grothendieck 1980, English Translation by Roy Lisker, Begun December 13, 2002.

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1. Topological tensor products and nuclear spaces.
2. Continuous and discrete duality (derived categories, “six operations”).
3. Riemann-Roch-Grothendieck Yoga (K-theory, relation with intersection theory).
4. Schemes.
5. Topos.
6. Etale and l-adic Cohomology.
7. Motives and motivic Galois group (Grothendieck categories).
8. Crystal and crystalline cohomology, yoga of “de Rham coefficients ”, “Hodge coefficients”. . ..
9. “Topological Algebra”: 1-stacks, derivators; cohomological topos formalism, as inspiration for a new homotopic algebra.
10. Tame Topology.
11. Algebraic anabelian geometry Yoga, Galois-Teichmüller theory.
12. Schematic or arithmetic point of view for regular polyhedras and regular configurations in all genera.

Each of these “new ideas” plunges deeply into the mathematical body and imposes on it a new systematic unity, or at least re-shapes our perspective on the different forms of unity it exhibits and enables us to trace new connections between them. The fact that we can distinguish these two opposite conceptions is as such significant. They are two forms of the creative productivity of mathematics. The first is that form in which it escapes us. The second is the form in which it gives ways of exercising control over its forms of expansion. We are able to recognize new trajectories and detect new relations, for example, the program of derived algebraic geometry, that consider polynomial equations up to homotopy. This is a new trajectory within a program. More precisely, it is a combination of schema theory and homotopy theory. Schema theory is re-worked from a homotopical perspective. The synthesis of both theories retains the power of each within a further unity. This allows a higher level viewpoint, permitting us to reinterpret both theories, and at the same time provides them with greater power. As a matter of fact, the unity as a synthesis of different elements, or different disciplines. Among the examples given above that would require immense development. We will be interested (only partially) in the theory of schemes (Hartshorne 1977).[6] We will proceed in four steps in order to explain the elementary concept of a scheme.

12.2.4 Example of Synthetic Unity: The Concept of Scheme

(a) We construct the space $Spec A$ associated to a ring A. As a set we define $Spec A$ to be the set of all prime ideals of A. We assume known the concept of ring and ideal and prime ideal. If $\mathfrak{a}$ is an ideal of A, we define the subset $V(\mathfrak{a}) \subseteq Spec A$ to be the set of all prime ideals which contain $\mathfrak{a}$. These concepts are purely

[6] Robin Hartshorne, *Algebraic Geometry*, Springer, New York, 1977.

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algebraic concepts. They refer to an important part of commutative algebra: the theory of ideals.

(b) Now we define a topology on $SpecA$ by taking the subsets of the form $V(\mathfrak{a})$ to be the closed subsets. We show that finite unions and arbitrary intersections of set of the form $V(\mathfrak{a})$ are again of that form. $V(\emptyset) = SpecA$, and $V(0) = SpecA$. You can see how algebra and topology form a specific unity. But this synthesis is not yet complete.

(c) The concept of a sheaf provides a systematic way of (Hartshorne 1977) [Grothendieck EGA I][7] discerning and taking account of local data. Sheaves are essential in the study of schemes. The concept of sheaf is another synthesis between algebra and topology. We give the definition.

Let X be a topological space; A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the data

(i) for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$ and
(ii) for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject to the conditions

(1) $\mathcal{F}(\emptyset) = 0$ where $\emptyset$ is the empty set
(2) ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$
(3) if $W \subseteq V \subseteq U$ are three open subsets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

A presheaf is a concept that is easy to express in the language of the categories that makes this unity of domains or disciplines appear: a presheaf is just a contravariant functor from the category $\mathfrak{Top}$ of topological spaces to the category $\mathfrak{Ab}$ of abelian groups.

If $\mathcal{F}$ is a presheaf on X we refer to $\mathcal{F}(U)$ as the section of the presheaf $\mathcal{F}$ over the open set U. Indeed we have to understand that we dispose a map s from U to $\mathcal{F}$, denoted as sections of the presheaf.

A sheaf is a presheaf satisfying some extra conditions. We will give only one condition in mathematical form.

If U is an open set, if $\{V_i\}$ is an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s_{|V_i} = 0$ for all i then $s = 0$.

The second condition is the condition that says that sections that coincide in the intersection of both open sets glue together in an unique section. It is the essential property of gluing that makes one pass from the local to the global. The different syntheses above syntheses are able to give a philosophical synthesis: the reflexive synthesis that allows us to know if a property can be globalized. This unity is beyond the unity between concepts or between different disciplines, it is a synthetic unity that "constructs" the globality of a property. There are many examplesof sheaves, such

[7]EGA I, Le langage des schémas. *Publ. Math. IHES* 4, 1960.

as the sheaf of continuous functions on a topological space, the sheaf of distributions etc. In this construction algebra and topology play a role at different levels.

(d) Let A be a ring. The *spectrum* of A is the pair consisting of the topological space $Spec A$ together with the sheaf of rings $\mathcal{O}$ defined above. To each ring A we have associated its spectrum $Spec A$, $\mathcal{O}$. This association is not complete. We would like that this correspondence/association is really synthetic unity which could be constructed as a conceptual, mathematical unity. If the ring A can be seen as a category, and it is also the case for the Spectrum we require this correspondence to be fonctorial.[8] The appropriate notion is the category of locally ringed spaces. So a *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. And next we must define what a morphism of ringed spaces consists of.

We get our last definition. An *affine scheme* is a locally ringed space $\mathcal{O}_X$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A scheme is a locally ringed space $\mathcal{O}_X$ in which every point has an open neighborhood U such that the topological space together with the restricted sheaf $\mathcal{O}_{X|U}$ is an affine scheme. We call X the *the underlying topological space* of the scheme $(X, \mathcal{O}_X)$ and $\mathcal{O}_X$ its *structure sheaf*.

This is a complicated unity, about which we shall make some remarks. The synthesis we have carried out makes it possible to make the various elements applies, in particular the topological element. But this one in turn plays within the algebraic control of the topological structure.

This description can be repeated for every theme developed by Grothendieck.

> To speak frankly these innumerable questions, notions, and formulations of which I've just spoken, indeed, the countless questions, concepts, statements I just mentioned, only make sense to me from the vantage of a certain "point of view" - to be more precise, they arise spontaneously through the force of a context in which they appear self evident: in much the same way as a powerful light (though diffuse) which invades the blackness of night, seems to give birth to the contours, vague or definite, of the shapes that now surround us. Without this light uniting all in a coherent bundle, these 10 or 100 or 1000 questions, notions or formulations look like a heterogeneous yet amorphous heap of "mental gadgets", each isolated from the other - and not like parts of a totality of which, though much of it remains invisible, still shrouded in the folds of night, we now have a clear presentiment. The fertile point of view is nothing less than the "eye", which recognizes the simple unity behind the multiplicity of the thing discovered. And this unity is, veritably, the very breath of life that relates and animates all this multiplicity.[9]

[8] For the concept of functor see Sect. 5.3.3(ii).

[9] Récoltes et Semailles Part I, The life of a mathematician. Reflection and Bearing Witness, Alexander Grothendieck, English translation by Roy Lisker, Begun December 13, 2002.

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12.3 This Architectonic Unity Takes Different Forms in the History of Mathematics. Three Philosophical Forms

12.3.1 Unity as Logical Unity or Operational Concept

We will build on Lautman to develop this philosophical conception and eventually to criticize the opposite conceptions. The first conception we appeal to here is founded on logical developments, like those proposed by Russell and Carnap. The second on Wittgenstein. As a matter of fact, this second conception explains that mathematical statements should be explained in terms of logical *operations*. Nevertheless, this approach is divorced from mathematical reality (which we also want to promote and analyze), and for this reason it was rejected by Lautman. Above all, he refuses the reduction of philosophy to the syntactic study of scientific utterances, and he rejects the reduction of philosophy to a role of clarification of propositions which intervene in "what is generally called the theory of knowledge". For example, propositions on space and time must be subject to criticism from the syntactic point of view.[10]

> Mathematical philosophy is often confused with the study of different logical formalisms. This attitude generally results in the affirmation of the tautological character of mathematics. The mathematical edifices which appear to the philosopher so difficult to explore, so rich in results and so harmonious in their structures, would in fact contain nothing more than the principle of identity. We would like to show how it is possible for the philosopher to dismiss such poor conceptions and to find within mathematics a reality which fully satisfies the expectation that he has of it.[11]

Lautman talks about the fading away of mathematical reality, and his judgement holds for Russell, Carnap and Wittgenstein. Alongside this logicist philosophical unity he excludes, Lautman retains two other conceptions of unity which are close to his own. From one side, mathematical reality can be characterized by the way one apprehends and analyzes its organization. From the other side, it can also be characterized in a more intrinsic fashion, from the point of view of its own structure. The first case was illustrated by Hilbert's position where he stressed the dominant role of metamathematical notions compared to those of the mathematical notions they serve to formalize. On this view, a mathematical theory receives its value from the mathematical properties that embody its structure in some generic sense. We recognize in this approach one (very influential) structural conception of mathematics. Indeed Hilbert substitutes for the method of genetic definitions the method of axiomatic definitions. He introduces new variables and new axioms, from logic to arithmetic and from arithmetic to analysis, which each time enlarge the area of consequences. For example, in order to formalize the analysis, it is necessary to be able to apply

[10] Lautman (2006, pp. 52–53) *Les mathématiques, les idées et le réel physique* Vrin, Paris. Introduction and biography by Jacques Lautman; introductory essay by Fernando Zalamea. Preface to the 1977 edition by Jean Dieudonné. Translated in Brandon Larvor *Dialectics in Mathematics*. Foundations of the Formal Science, 2010.

[11] Lautman (2006) *ibid.*

the axiom of choice, not only to numerical variables, but to a higher category of variables, those in which the variables are functions of numbers.[12] The mathematics is thus presented as successive syntheses in which each step is irreducible to the previous stage.

Moreover, it is necessary to superimpose a metamathematical approach on this formalized mathematical theory which takes it as an object of analysis from the point of view of non-contradiction and its completion.[13,14,15] When we recall the Hilbertian point of view, we see that a duality of plans between formalized mathematics and the metamathematical study of this formalism entail the dominant role of metamathematical notions in relation to formalized mathematics.

Lauman quotes Hilbert *Gesammelte Abhandlungen* t. III, p. 196 sq.[16] and Paul Bernays, *Hilberts Untersuchungen über die Grundlagen der Arithmetik.*[17]

In this structural, synthetic conception,[18] mathematics is seen—if not as a completed whole—then at least as a whole within which theories are to be regarded as qualitatively distinct and stable entities whose interrelationships can in principle be thought of as completely specifiable,

What is the meaning of Hilbert's structuralism? Lautman[19] takes the example of the Hilbert space. The consideration of a purely formal mathematics leaves the place in Hilbert to the dualism of a topological structure and functional properties in relation to this structure. The object studied is not the set of propositions derived from axioms, but complete organized beings having their own anatomy and physiology. For Lautman the Hilbert space is "defined by axioms which give it a structure appropriate to the resolution of integral equations. The point of view that prevails here is that of the synthesis of the necessary conditions and not that of the analysis of the first conditions"[20]

As the second conception we recognize a more dynamic diachronic picture of the interrelationships, which sees each theory as coming with an indefinite power of expansion beyond its limits bringing connections with others, of a kind which

[12] *ibid.*, Lautman (2006, p. 130).

[13] Lautman means completeness in the sense of completion. The system is said to be completed if any proposition of the theory is either demonstrable or refutable by the demonstration of its negation. The property of completion is said to be structural because its attribution to a system or a proposal requires an internal study of all the consequences of the considered system.

[14] Recall that I am analyzing the philosophical architectonic unity of mathematics. This was illustrated by Hilbert's position. He stressed the dominant role of metamathematical notions compared to those of the mathematical notions they serve to formalize. On this view, a mathematical theory receives its value from the metamathematical properties that embody its structure in some generic sense. We recognize in this approach one (very influential) structural conception of mathematics.

[15] Lautman (2006, p. 30).

[16] David Hilbert, *Gesammelte Abhandlungen*, Verlag Julius Springer Berlin, 1932.

[17] Paul Bernays, *Hilberts Untersuchungen über die Grundlagen der Arithmetik*, Springer, 1934.

[18] Lautman, *Essai sur les notions de structure et d'existence*, Hermann, Paris 1937: the structural point of view to which we must also refer is that of Hilbert's metamathematics etc.

[19] Lautman (2006, pp. 48–49).

[20] Lautman, *ibid.*

confirms the unity of mathematics, especially from the standpoint of mathematical epistemology.

In Hilbert's metamathematics one aims to examine mathematical notions in terms of notions of non-contradiction and completion. This ideal turned out to be unattainable. Metamathematics can consider the idea of certain perfect structures, possibly realized by effective mathematical theories. Lautman wanted to develop a framework that combines the fixity of logical concepts and the development that gives life theories.

12.3.2 Dialectics

Lautman (in a third conception, the one he defends) wanted to consider other logical notions that may also be connected to each other in a mathematical theory such that solutions to the problems they pose can have an infinite number of degrees. On this picture mathematics set out partial results, reconciliations stop halfway, theories are explored in a manner that looks like trial and error, which is organized thematically and which allows us to see the kind of emergent linkage between abstract ideas that Lautman calls *dialectical*.

Contemporary mathematics, in particular the development of relations between algebra group theory and topology appeared to Lautman to illustrate this second—our words—"labyrinthine" model of the dynamic evolving unity of mathematics, structured around oppositions such as local/global, intrinsic/extrinsic, essence/existence. It is at the level of such oppositions that philosophy intervenes in an essential way.

> It is insofar as mathematical theory supplies an answer to a dialectical problem that is definable but not resolvable independently of mathematics that the theory seems to me to participate, in the Platonic sense, in the Idea with regard to which it stands as an Answer to a Question.

(Lautman 2006, p. 250)[21]

12.3.3 Philosophical Choices

Lautman seeks to study specific mathematical structures in the light of oppositions such as continuous/discontinuous, global/local, finite/infinite, symmetric/antisymmetric. Brendan Larvor[22] remarks that in *New Research on the Dialectical Structure of Mathematics* Lautman offers a slightly different list of dialectical poles: "whole and parts situational and intrinsic properties, basic domains and objects

[21] Brendon Larvor, *Albert Lautman: Dialectics in Mathematics*, Foundations of formal Science, 2010.

[22] Brendan Larvor, *Albert Lautman*, *ibid.*

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defined on these domains, formal systems and their models etc."[23] In his book (Lautman 2006), there is a chapter about "Local/global". He studies the almost organic way that the parts are constrained to organize themselves into a whole and the whole to organize the parts. Lautman says "almost organic": one thinks of this expression in the following way. There exists an "organic" unity within mathematics, reminiscent of biological systems, as Alain Connes amongst many others has noted (see above). We want to develop this recognition further below by making use of the notion of reflexivity. In his chapter on extrinsic and intrinsic properties with title "Intrinsic properties and induced properties", Lautman examines whether it is possible to reduce the relationships that some system maintains with an ambient medium to properties inherent to this system. In this case he appeals to classical theorems of algebraic topology. More well-known is his text on "an ascent to the absolute", in which an analysis of Galois theory, class field theory, and the uniformization of algebraic functions on a Riemannian surface is presented. Lautman wanted to show how opposite philosophical categories are incarnated in mathematical theories. Mathematical theories are data for the exploration of ideal realities in which this material is involved.

12.3.4 Concerning On the Unity of the Mathematical Sciences

This is the first of Lautman's two theses. It takes as its starting point a distinction that Hermann Weyl made in his 1928 work on group theory and quantum mechanics.[24] Weyl distinguished between "classical" mathematics, which found its highest flowering in the theory of functions of complex variables, and the "new" mathematics represented[25] by the theory of groups and topology (Lautman 2006, p. 83). For Lautman, the classical mathematics of Weyl's distinction is essentially analysis,[26] that is, the mathematics that depends on some variables tending toward zero, convergent series, limits, continuity, differentiation and integration. It is the mathematics of arbitrary small neighborhoods, and it reached maturity in the nineteenth century. And, Brendan Larvor continues, the 'new mathematics of Weyl's distinction is global': it studies structures of "wholes".[27] Algebraic topology, for example, considers the properties of an entire surface (how many holes) rather than aggregations of neighborhoods.

Having illustrated Weyl's distinction, Lautman re-draws it.[28]

[23]Lautman (2006).

[24]Hermann Weyl, *Gruppentheorie and Quantenmechanik*, Hirzel, Leipzig, 1928.

[25]Larvor (2010).

[26]Brendan Larvor, *ibid.*

[27]Lautman (2006, p. 84).

[28]Larvor (2010), Lautman, 2005, p. 196.

> In contrast to the analysis of the continuous and the infinite, algebraic structures clearly have a finite and a discontinuous aspect. Through the elements of a group, a field or an algebra (in the restricted sense of the word) may be infinite, the methods of modern algebra usually consists in dividing these elements into equivalence classes, the number of which is, in most applications, finite.[29]

The chief part of Lautman's "unity" thesis is taken up with four examples in which theories of modern analysis [see Brendan Larvor] depends in their most intimate details on results and techniques drawn from the "new" (Larvor 2010) algebraic side of Weyl's distinction. Algebra comes to the aid of analysis. That is, dimensional decomposition in function theory; non-Euclidian metrics in analytic function theory; non commutative algebras in the equivalence of differential equations; and the use of finite, discontinuous algebraic structures to determine the existence of the function of continuous variables.[30]

Lautman transforms a broad historical distinction (between the local, analytic, continuous and infinitistic mathematics of the nineteenth century, and the 'new' global, synthetic, discrete and finitistic style) into a family of dialectical dyads (local/global, analytic/synthetic, continuous/discrete, infinitistic/finitistic. These pairs find their content in the details of mathematical theories (Larvor 2010), that, though they belong to analysis, sometimes employ a characteristically algebraic point of view.

12.4 Mathematical Reflexivity

The topic of this section is the study of the development of forms of *reflexivity* in mathematics, which imply the history of the concept of space and the history of several disciplines. Mathematical activity involves, as an essential aspect, examining concepts, theories or structures through (the lens of) other concepts theories and structures, which we recognize as reflecting them in some way.

There are many ways to understand the notion of reflexivity in mathematical practice. One such way is illustrated by the stacking of algebraic structures, groups, rings, fields, vector space, modules, etc. Each level is the extension of the previous one—a vector space for example is a certain kind of module. The extension here consists in adding a property or a law. This imposition of additional structure brings a new perspective on the initial structure. A second way involves the addition of some property coming from another domain altogether, as seen in the notions of topological group, Lie group, differential or topological field. This synthesis also yields a new view of the initial structure. This is reflexivity in the weak sense. The effect of such new syntheses makes up much of the history of mathematics. But one has synthesis also between structures or concepts.

[29] Lautman (2006), pp. 86–87.

[30] Lautman (2006, p. 87).

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This other kind of the reflexivity includes the case where one discipline, for example algebra, reflects some concepts or some structures from another discipline. For example in algebraic topology, algebraic concepts and methods are used to translate and to control some topological properties. The same holds for algebraic geometry. And it can happen that algebraic topology and geometry themselves cross-fertilize by means of such reflexive interactions. All the phenomena of translation of one discipline in to another also illustrate such forms of reflexivity. This is a local manifestation of the fact that mathematics is permeated with such "reflecting surfaces".

It is possible also to construct the history of one concept, for example, the concept of number or of space viewed from this standpoint. Gilles-Gaston Granger, a French philosopher of mathematics, says that these concepts are "natural". But they are also the most opaque.[31]

Notice that the history of the concept of space through the concept of a manifold involves the intersection of multiple disciplines and the development of multiple forms of *reflexivity*.

12.4.1 The Concept of a Manifold

In the case of space, there was a long process whereby a deepening reflection on the concept of surface was produced in mathematics. Along the way, such a concept as that of variety was revealed. The concept of variety arose as a geometrical *reflection* on the concept of surface: First came the notion of an abstract surface parameterized by coordinates, then that of abstract place covered by topological opens (maps, atlas) in relation to an ambient space. These notions were extracted from such "reflexive" contexts as autonomous concepts which could be seen as defining a new kind of mathematical entity.

This extraction involved abstraction from the concept of surface, an abstraction which at the same time brought a change of point of view on the earlier concept: one passed from a concept defined via coordinates to one resting on parameters. That passage was effected by a reflection on the sense of using coordinates. One can understand that a surface is nothing but the different forms of the variation of its coordinates. And when one speaks in terms of maps and atlases the concept is further deeply reworked as was achieved by Hermann Weyl in his *Concept of Riemann surface* 1912.[32] This new entity now acts as the carrier of topological properties, and manifolds come to be seen as autonomous entities and indeed as a fundamental concept. The act whereby we obtained a surface is geometrically displaced, so to speak, and in this act of displacement the entity to which it is related is re-defined. The notion of a variety is likewise designated in functional terms: it is the range of variation of the values of certain functions. Functions can now reflect their nature

[31] Gilles-Gaston Granger, *Formes opérations, objets* Paris, Vrin, 1994, pp. 290–292.

[32] Hermann Weyl, *The Concept of a Riemann Surface*, Addison and Wesley, 1964, First Edition *die Idee der Riemanschen Fläche*,Teubner, Berlin, 1912.

by means of this new entity. A manifold becomes the support and mirror for the properties of functions that are defined on it. These functions with their properties constitute the new objects we should consider as new basis and point of departure for a further stage of geometrization.

We would like to give a particularly striking example. We remind that a functor **F** from a category C to another category D is a structure-preserving function from C to D. Intuitively, if C is een as a network of arrows beteween objects, then **F** maps that network onto network of arrows of D. Every category C has an identity functor $1_A : C \to C$ which leaves the objects and arrows of C unchanged, and given functors, $\mathbf{F} : C \to D$ and $\mathbf{G}D \to E$ there is a composite $\mathbf{G} \circ \mathbf{F} : C \to E$. So it is natural to speak of a category of all categories, which we call **CAT**, the objects of which are all the categories and the arrows of which are all functors. And Colin McLarty asks whether **CAT** is a category in itself. His answer is to treat **CAT** as a regulative idea; an inevitable way of thinking about categories and functors, but not a strictly legitimate entity.[33,34] In a not so formal sense we can get a notion of common foundation for mathematics in the elementary notions that constitute categories. The author believes, in fact, that the most reasonable way to arrive at a foundation meeting these requirements is simply to write down axioms descriptive of properties which the intuitively-conceived category of all categories has until an intuitively adequate list is attained; that is essentially how the theory described below was arrived at.[35] Thus our notion of space changes status, it becomes an intelligible object in itself, and that is why it can provide a reflexive context in which to reconceptualize the previous notion of a surface. At the same time, the act of measuring can be considered as such and made the object of study as a structure within the mathematical body. The concept of a metric on a manifold makes possible this new reflexion. Any such expression of magnitude can be reduced to a quadratic form, and thereby expresses the most general law that defines the distance between two infinitely near points of a variety.

This entity in turn enables us to construct new spaces: we can now define the notions of algebraic manifold, topological manifold, differentiable manifold, analytic manifold, arithmetic manifold. In this way we are given the means to pass from one discipline to another. This passage between formerly separated disciplines involved both an upward movement (in the formation of the concept of manifold) and horizontal and synthetic extension of concepts (across several domains and disciplines).

[33]Immanuel Kant, *Kritik der reinenVernunft*, Hartnoch Transl. N. Kemp Smith (1929) as *Critique of Pure Reason*, Mcmillan.

[34]Colin McLarty, *Elementary Categories, Elementary Toposes*, Clarendon Press Oxford, 1992, p. 5 "Compare the self, the universe and God in Kant 1781".

[35]William Lawvere, The category of categories as a foundation of mathematics by, *Proceedings of the Conference on Categorical Algebra, La Jolla Calif.* 1965, pp. 1–20, Springer Verlag, New York, 1966.

12.5 Reflexivity and Unity

One of the most powerful tools we use to explain the reflexivity is the concept of topos, and moreover the concept of Grothendieck's topos.

12.5.1 Prerequisites for Understanding the Search for the Unity of Mathematics According to Grothendieck

I distinguish three prerogatives that underlie the arguments of Grothendieck.

(a) The unity of mathematics according to Grothendieck is that of the discrete and the continuous, and the structure of mathematics must be able to account for it.

(b) Three aspects of mathematical reality are traditionally distinguished. Number, or the arithmetical aspect; size, or the analytical aspect; form, or the analytical aspect.[36] Grothendieck took an interest in form as embodied in structures.

> This means that if there is one thing in mathematics that has always fascinated me more than any other, it is neither number nor size, but always form. And among the thousand and one faces that form chooses to reveal itself to us, the one that has fascinated me more than any other and continues to fascinate me is the hidden structure in mathematical things.[37]

(c) Grothendieck adopts a resolutely realistic attitude.

> The structure of a thing is by no means something we can invent. We can only patiently update, humbly get to know it, "**discover**" it. If there is inventiveness in this work, if we happen to be a blacksmith or indefatigable builder, it is not to "shape" or to build structures ... It is to **express** as faithfully as we can these things that we are discovering and probing, this reluctant structure to indulge ...

The sequel of the quotation describes both tasks

> Inventing language capable of expressing more and more finely the intimate structure of the mathematical thing and ... constructing, with the aid of this language, progressively and from scratch, the theories which are supposed to account for what has been seen and apprehended.

(RS 1985)[38]

> One might say that Numbers are what is appropriate for grasping the structure of discontinuous or discrete aggregates. These systems, often finite, are formed from "elements" or "objects" conceived as isolated with respect to one another. "Magnitude" on the other hand is the quality, above all, susceptible to "continuous variation", and is most appropriate for grasping continuous structure and phenomena: motion, space, varieties in all their forms, force, field, etc. Therefore arithmetic appears to be (over-all) the science of discrete structures while analysis is the science of continuous structures.[39]

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[36] Mathieu Belanger, *La vision unificatrice de Grothendieck: au-delà de l'unité (méthodologique?) de Lautman* Philosophiques vol 37 Numéro 1–2010.

[37] Grothendieck, *Récoltes et Semailles,* 1985, *Reaping and Sowing* my translation.

[38] *ibid.*, my translation.

[39] A. Grothendieck, Récoltes et Semailles, traduction Roy Lisker p. 66.

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It is therefore necessary to understand that the point of view of number is used for the discrete structure, whereas the point of view of magnitude is used to grasp the structure of the continuum. According to Grothendieck "arithmetic is the science of discrete structures and analysis is the science of continuous structures",[40] and for him, geometry intersects both the discrete structures and the continuous structures. The study of geometrical figures could be done from two distinct points of view. First, the combinatorial topology in Euler's sequence was linked to the discrete properties of the figures. Second, geometry (synthetic or analytic) examined the continuous properties of the same figures. It was based in particular on the idea of size expressed in terms of distances. Geometry studied both the discrete and the continuous, but distinctly.[41] The development of abstract algebraic geometry in the 20th century inaugurated a renewal of the aspect of form by imposing a single point of view that directly participates in both the study of discrete structures and the study of continuous structures.

12.5.2 Prerequisites to the Search for Unity as Implementation of Reflexivity

The search for unity through the creation of a new discipline consists in seeing how analysis can be reflected in arithmetic, and how arithmetic can be reflected in analysis. Whenever a concept of one discipline is enlightened by another, it is analyzed by the other: by associating a concept of the concerned discipline and a form of abstraction. It refers this form to the first discipline. This reflexivity has taken place in the new algebraic geometry of Grothendieck in a double manner, or in a mirror with two faces: one face for arithmetic and the other for analysis, Grothendieck called it "arithmetical geometry".

12.5.3 The rôle Played by Weil's Conjectures

Working on abstract algebraic geometry, the great French mathematician André Weil formulated four conjectures concerning the zeta function of algebraic manifolds on finite fields. We cannot expose the very great complexity of these conjectures. It is sufficient to know for our purposes that the very great generality of these conjectures and their difficulty was due to the fact that they required the application of topological invariants to algebraic varieties. According to Grothendieck, Weil's conjectures required the construction of a bridge between continuous structures and discrete structures. The Weil conjectures served as a guide to the elaboration of the

[40] *Ibid.*

[41] Mathieu Belanger [Belanger p. 15 online].

new geometry. We can see unity and reflexivity in the context of the new arithmetical geometry.

> It may be considered that the new geometry is above all else a synthesis between these two adjoining and closely connected but nevertheless separate parts: the arithmetical world in which the so-called spaces live without the principle of continuity, and the world of continuous quantity, that is, space in the proper sense of the term, accessible through analysis (and for that very reason) accepted by him as worthy to live in the mathematical city. In the new vision, these worlds, formerly separated, form but one.[42]

12.5.4 Reflecting Space to Produce a New Topological Concept

The traditional concept of space does not have the flexibility required by the topological invariants of arithmetic geometry. However, no concept of space was more general than that which prevailed in the 1950s.

12.5.5 The Generality of the Topological Space

A topology is considered as the most stripped-down and therefore the most general structure available to a space. Let E be any set. Constructed from a family of sets, topology $\mathcal{P}(\mathcal{P}(E))$ chooses those that respect a stability for finite union set operations (for the definition of topological open sets) and for any intersection. This is a first level of reflexivity. Indeed, the operation of taking the parts of a set is redoubled on itself, and makes it possible to choose, according to a rule, certain subsets. It is also a way of analyzing the subsets of a set. The iteration is identified with a form of reflexivity. The concept of topological space encompasses all other space concepts.

We see here the intervention of set concepts to give the notion of space a form that goes beyond its static bases thanks to the set operations which possess a structure of algebra. But the most flexible spatial structure available to mathematicians was not sufficient for the problem raised by Weil's conjectures. The cornerstone of the new geometry therefore had to be a concept of space allowing one to go beyond the maximum generality of the concept of traditional topological space.

12.5.6 The Concept of Topos According to Grothendieck

The concept of topos provides maximum generality. It allows us to form a unit and realizes a form of reflexivity.

[42]Reaping and Sowing, 1985, my translation.

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12.5.7 Back to Concept of Sheaf

Grothendieck replaces the lattice of open subsets, which defines the structure of a topological space in the traditional sense. He uses the notion of sheaf, which we have defined above. A sheaf is a mathematical concept allowing one to define a mathematical structure defined locally on a space X by a process of restriction and gluing. Some definitions are in order

(i) A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cap Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

(ii) Let k be a fixed algebraically closed field. We define affine n-space over k denoted $\mathbf{A}^n$ to be the set of all n-uples of elements of k. An *affine algebraic variety*, (or simply *affine variety*) is an irreducible closed subset of $\mathbf{A}^n$ with the induced topology from the topology of $\mathbf{A}^n$. An open subset of an affine variety is a *quasi-affine variety*.

(iii) A function $f : Y \to k$ is regular at a point $P \in Y$ if there is an open neighborhood U with $U \subseteq Y$ and polynomials $g, h \in A = k[x_1, \dots x_n]$ such that h is nowhere zero on U and $f = g/h$ on U. $A = k[x_1, \dots x_n]$ is the polynomial ring on a field k.

(iv) Let X be a variety over the field k. For each open set $U \subseteq X$ let $\mathcal{O}(U)$ the ring of regular functions from U to k. $\mathcal{O}$ verify the conditions of presheaf and of sheaf. These functions form a ring because they verify the ring's operations.

As we remarked we can consider the unity that ring structure gives functions and the way these functions are reflected by means of algebraic structure.

One can define the sheaf of continuous real-valued functions, the sheaf of differentiable functions on a differentiable manifold, or the sheaf of holomorphic functions on a complex manifold.

If we consider the lattice of open subsets of a topological space X, (we can also denote with $\mathcal{O}(X)$), the real-valued functions $f : U \in \mathbb{R}$, the restrictions of $f_{|V}$ on the open subsets $V \subset U$. By means of correct choice of V_i it is possible to reconstruct the function from its restrictions.

12.5.8 The Language of Category Theory

(i) Grothendieck considers the totality of sheafs on a topological space; It is the remarkable generative effect produced by this approach. All the sheafs on a topological space X form a category, denoted $Sh(X)$. This category is essential because it makes it possible to find the topological structure of space, that is to say the lattice of the open spaces $\mathcal{O}(X)$.

The topological structure is in fact determined by the category of the sheafs, which is much more flexible than the topological structure. It possesses the flexibility sought to transcend the apparent generality of the concept of traditional topological space.

12.5.9 Grothendieck Topology

In the usual definition of a topological space and of a sheaf on that space, one uses the open neighborhoods U of a point in a space X. Such neighborhoods are topological maps: $U \to X$ which are injective. For algebraic geometry it turned out that it was important to replace these injections (inclusions) by more general maps $Y \to X$ which are no necessarily injective. It extends the application by relieving constraints and preserving only the application whose source is any object of the category. The idea of replacing inclusions $U \to X$ by more general maps $U' \to X$ led Grothendieck to define "the open covers" of X.

We see that Grothendieck systematically considers the point of view of applications (morphisms) instead of objects (sets).

(i) covering families

Let **C** be a category and let C be an object in **C**. Consider the indexed families

$$S = \{f_i : C_i \to C | i \in I\}$$

and suppose that for each object C of **C** we have a set

$$K(C)\{S, S', S'', \dots ,\}$$

of certain such families called the *coverings* of C under the rule K. Thus for these coverings we can repeat the usual topological definition of a sheaf.

(ii) category equipped with covering families

To introduce a general notion of a category equipped with *covering families* we first use a functor. A functor—as we know—is a map (morphism) from a category to another category. For example, there is a functor from **C** any category to the category of sets, **Set**. We take the opposite category denoted $\mathbf{C}^{op}$. It is a category for which the maps (morphisms) are reversed with respect to the morphisms of the starting category. There is a functor from the category $\mathbf{C}^{op}$ to the category **Set**.

Thus we dispose the functor category denoted $\mathbf{Set}^{\mathbf{C}^{op}}$. Let us note the rise in abstraction, first the categories (objects and arrows) then the opposite categories, then the functors, passage from one category to another, and finally the category whose objects are the functors. We do not define "natural" applications that are not necessary for us. The use of a functor is necessary to see at the same time the passage from one category to another, which thus forges a possible unity and reflection reflected from one category to another. Each time this unity and this reflection take on a different meaning.

(iii) sieve

Grothendieck defines a notion of topology anchored in the category theory. It is also more general and more flexible than the traditional set of concepts. It uses the concept of a sieve. A sieve S may be given as a family of morphisms in $\mathbf{C}$ all with codomain C, such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense; in other words S is a right ideal. If S is a sieve on C and $h : D \to C$ is any arrow to C then

$$h^*(S) = \{g | cod(g) = D, h \circ g \in S\}$$

is a sieve on D. A sieve is a conceptual tool that makes it possible to gather arrows that are composed. Intuitively it is a collection of arrows that is "to hang" one to the other.

(iv) A Grothendieck topology on a category $\mathbf{C}$ is a function J which assigns to each object C of $\mathbf{C}$ a collection $J(C)$ of sieves on C, in such a way that

(a) the maximal sieve $t_C = \{f | cod(f) = C\}$ is in $J(C)$;
(b) (stability axiom) if $S \in J(C)$ then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;
(c) (transitivity axiom) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S then $R \in J(C)$.

A Grothendieck topology is at a higher level a reflexion of topology. Here is a quotation by F. William Lawvere.

> A Grothendieck topology appears most naturally as a modal operator, of the nature "it is locally the case of".[43]

Grothendieck topology chooses some sieves. It is first and foremost a way of making the covering families respecting a stability of operative composition on the objects that it targets. If $S \in J(C)$, one says that S is a *covering sieve* or that S *covers* C (or, if necessary, that S J- covers C). Reflexivity here takes the form of the dynamic establishment of the conditions under which one can construct a topology.

In the case of an ordinary topological space, one usually describes an open cover U as just a family, $\{U_i, i \in I\}$ of open subsets of U with union $\bigcup U_i = U$. Such a family is not necessarily a sieve, but it does generate a sieve-namely, the collection of all those open $V \subseteq U$ with $V \subseteq U_i$ for some U_i. [Saunders Mac Lane, Ieke Moerdijk

[43] See below Sect. 6.1.7.

1992].[44] (Informally V goes through the sieve if it fits through one of holes U_i of the sieve).[45]

(v) Site

A Grothendieck topology has a basis from which we give elements and properties.

A basis for a Grothendieck topology on a category C with pullbacks (that is with some sort of inverse image) is a function K which assigns to each object C a collection $K(C)$ consisting of families of morphisms with codomain C such that

(i') if $f : C' \to C$ is an isomorphism, then $\{f : C' \to C\} \in K(C)$;
(ii') if $\{f_i : C_i \to C | i \in I\} \in K(C)$ then for any morphism $g : D \to C$ the family of pullbacks $\{\pi_2; C_i \times_C D \to D\}$ is in $K(D)$;
(iii') if $\{f_i : C_i \to C | i \in I\} \in K(C)$, and if for each $i \in I$ one has a family $\{g_{ij} : D_{ij} \to C_i | j \in I_i\} \in K(C)$ then the family of composites $\{f_i \circ g_{ij} : D_{ij} \to C | i \in I, j \in I_i\}$ is in K(C) .

Condition (ii') is again called the stability axiom, and (iii') the transitivity axiom. The pair $(\mathbf{C}, K)$is also called a *site* and the elements of the set $K(C)$ are called *covering families* or *covers* for this site. Covering families (we can denote *Cov*) are also called a *pretopology*. The pair (C, Cov) is a site.

The definition of the base pushes the notion of stability very far.

12.5.10 Grothendieck Topos

A Grothendieck topos is a category which is equivalent to the category $Sh(\mathbf{C}, J)$ of sheaves on some site $\mathbf{C}, J$. Or in other words let be a *stack* or with another name a *presheaf* of sets over a category $\mathcal{C}$: it is a (contravariant) functor $F : \mathcal{C} \to \mathbf{Set}$.

We need to add the following remarks. The category $\mathbf{St}(\mathcal{C})$ of all stacks over $(\mathcal{C})$ is equivalent to the topos $\mathbf{Set}^{\mathcal{C}^{op}}$. This is an elementary topos like **Set** or **Finset** and other simple topoi. They are, generally speaking, some domain where mathematics can be developed, roughly speaking, without problems. We can do mathematics without thinking about it. Grothendieck topoi are more complicated.

We need to consider the subcategory of $\mathbf{St}(\mathcal{C})$ generated by those objects that are sheaves over the site $(\mathcal{C}, Cov$. It will denoted $\mathbf{Sh}(Cov)$. A Grothendieck topos is, by definition, any category that is equivalent to one of the form $\mathbf{Sh}(Cov)$.

[44] Saunders Mac lane, Jeke Moerbijk, *Sheaves and Geometry*, Springer, New York, 1992, pp. 110, 111.
[45] Mac Lane, Moerbijk, *ibid.*

12.6 Return to the Question of the General Unity of Mathematics and of the Reflexivity

Topoi theory is the way that Grothendieck constructed in order to find the solution of the problem of unity of the discrete and the continuous to resolve Weil's conjectures.

> The idea of topos encompasses, in a common topological intuition, the traditional (topological) spaces embodying the world of continuous magnitude, the (supposed) "spaces" or "varieties" of impenitent algebraic geometricians, as well as innumerable other types of structures, which until then had seemed irrevocably bound to the "arithmetical world" of "discontinuous" or "discrete" aggregates.[46]

The topoi tool allowing us to apply the topological invariants to an algebraic variety on a finite field makes the geometry a bridge between the arithmetical and analytical aspects of the mathematics, that is to say between the discrete and the continuous.

> It is the theme of the topos... which is this "bed" or "deep river" where geometry, algebra, topology and arithmetic, mathematical logic and the theory of categories come together, the world of continuous and that of 'discontinuous' or discrete structures.[47,48]

12.6.1 Brief Considerations on Topoi, Apropos of Reflexivity and Unity

12.6.2 From Grothendieck Topos to "Elementary" Topos

We have recalled the more general notion of coverings in a category (Grothendieck topology), the resulting "sites" as well as the topos formed as the category of all sheaves of sets on such a site. Then, [Mac Lane and Morbijk] said, in 1963, Lawvere embarked on the daring project of a purely categorical foundation of all mathematics, beginning with an appropriate axiomatization of the category of sets, thus replacing set membership by the composition of functions. This replacement is an essential movement of reflexivity, a transition to dynamic operations that transform any static basic link in set theory. Lawvere soon observed that a Grothendieck topos admits basic operations of set theory as the formation of sets Y^X of functions (all functions from X to Y) and of power sets $P(X)$ (all subsets of X). Lawvere and Tierney discovered an effective axiomatization of categories of sheaves of sets (and in particular, of the category of sets) via an appropriate formulation of set-theoretic properties. They defined, in an elementary way, free of all set-theoretic assumptions, the notion of an

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[46] Grothendieck, 1985, *Reaping and Sowing*, my translation.

[47] Grothendieck, 1985 *ibid.*

[48] Olivia Caramello developed a deep and powerful work on the "topos-theoretic background, and on the concept of a bridge" see "the bridge-building technique" in Olivia Caramello, 'Topos-theoretic background" IHES, September, 2014.

"elementary topos". They yield a final axiomatization of "beautiful and amazing simplicity" [Mac Lane and Moerdijk, p. 3]. An elementary topos is a category with finite limits, function objects Y^X for any two object Y and X, and a power object $P(X)$ for each object X; they are required to satisfy some simple basic axioms, like first-order properties of ordinary function sets and power sets in naive set theory. A limit is defined by means of a diagram (consisting of objects c in a category $\mathcal{C}$ together with arrows $f_i c \to d_i$), called a *cone*, that makes arrows to commute. A limit is a cone $\{f_i : c \to d_i\}$ with the property that for any other cone $\{f_i' : c' \to d\}$ there exists exactly one arrow $f' : c' \to c$ that makes both cones commute when composed.

Every Grothendieck topos is an elementary topos but not conversely. Lawvere's basic idea was that a topos is a "universe of sets". Intuitionistic logic, and the mathematics based on it, originated with Brouwer's work on the foundations of mathematics at the beginning of the twentieth century. He insisted that all proofs be constructive. That means that he did not allow proof by contradiction and hence that he excluded the classical *tertium non datur*. Heyting and others introduced formal system of intuitionistic logic, weaker than classical logic.

To understand this point let us make the following remark. In a topological space the complement of an open set U is closed but not usually open, so among the open sets the "negation" of U should be the interior of its complement. This has the consequence that the double negation of U is not necessarily equal to U. Thus, as observed first by Stone and Tarski, the algebra of open sets is not Boolean, but instead follows the rules of the intuitionistic propositional calculus. Since these rules were first formulated by A. Heyting, such an algebra was called a Heyting algebra.

Subobjects (defined below) in a category of sheaves have a negation operator which belongs to a Heyting algebra. Moreover—we follow [Mac Lane and and Moerbijk]—there are quantifier operations on sheaves, which have exactly the properties of corresponding quantifiers in intuitionistic logic. This leads to the remarkable result, that the "intrinsic" logic of a topos is in general intuitionistic. There can be particular sheaf categories, where the intuitionistic logic becomes ordinary (classical) logic. An arbitrary topos can be viewed as an *intuitionitistic universe of sets*.[49]

12.6.3 Some Brief Remarks on Benabou-Mitchel and Kripke-Joyal Languages

Mathematical statements and theorems can be formulated with precision in the symbolism of the standard firs-order logic. As Mac Lane and Moerdijk remind.[50] There

[49]This does not implies a revision of mathematics but the following position. There are structural principles of demonstration that most mathematicians use when demonstrating. These principles, if used alone, define a constructive or intuitionistic mathematics. It has structural rules, models, an essential notion of context, soundness, etc. It can be shown that the excluded middle can not be deduced from it etc. the job is to show if this logic is sufficient or to specify what of the classical logic should be available to do some demonstrations.

[50]Saunders Mac Lane and Ieke Moerdijk, *Sheaves in Geometry and Logic A firs introduction to Topos Theory*, p. 296 sq.

are at least four objectives for occasional such formulations (say, theorems of interest) as follows:

(1) They provide a precise way of stating theorems.
(2) They allow for a meticulous formulation of the rules of proof of that domain, by stating all "the rules of inference" which allow in succession the deduction of (true) theorems from the axioms in the domain.
(3) They may serve to describe an object of the domain—a set, an integer, a real number-as the set of all so and so's, thus in the language of natural numbers.
(4) They make possible a "semantics" which provides a description of when a formula is "true" (that is universally valid). Such a semantics (in terms of Mac Lane and Moerdijk) in terms of some domains of objects assumed to be at hand.

As in point (3), it is showed that formulas $\phi(x)$ in a variable x of the Mitchell-Benabou can be used to specify objects of $\mathcal{E}$ ($\mathcal{E}$ any topos) in expression of the form

$$\{x|\phi(x)\}$$

-in the fashion common in set theory. This shows how a topos behaves like a "universe of sets". One can for example, mimic the usual set-theoretic constructions of the integers, rationals, reals, and complex numbers and so construct in any topos with a natural numbers object, the object of integers, rationals, reals, ... Mac Lane and Moerdijk also show how the work of Beth and Kripke, in constructing a semantics for intuitionistic and modal logics, can also provide a semantics for the Mitchell Benabou language of a topos $\mathcal{E}$. In practice this means that one can perform many set-theoretic constructions in a topos and define objects of $\mathcal{E}$ as in (1); however- this is important—in establishing properties of these objects within the language of the topos *one should use only constructive and explicit arguments*.

12.6.4 An example, Preliminaries for Its Explanation

I cannot specify the language. I limit myself to giving essential features of this language. It can conveniently be used to describe various objects of $\mathcal{E}$.

I need firstly to define what a classifier consists of. In a category **C** with finite limits a subobject classifier is a monic (monomorphism) ($\equiv$ an injection), $1 \to \Omega$ such that every monic $S \to X$ in **C** there is a unique arrow ϕ which, with the given monic, forms a pullback square

$$\begin{array}{ccc} S & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow{\scriptstyle true} \\ X & \xrightarrow[\phi]{} & \Omega \end{array} .$$

This amounts to saying that the subobject functor is representable. In detail by simplifying in Moerdijk's fashion a subobject of an object X in any category $\mathbf{C}$ is an equivalence class of monics $m : S \to X$ to X. By a familiar abuse of language we say that the subobject *is* S or m meaning always the equivalence class of m.

A *pullback* or *fibered product* is the following. Given two functions $f : B \to A$ and $g : C \to A$ between sets, one may construct their fibred product as the set

$$B \times_A C = \{(b, c) \in B \times C | f(b) = g(c)\}.$$

Thus $B \times_A C$ is a subset of the product, and comes equipped with two *projections* $\pi_1 : B \times_A C \to B$ and $\pi_2 : B \times_A C \to C$ which fit into a commutative diagram

$$\begin{array}{ccc} B \times_A C & \xrightarrow{\pi_2} & C \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & \Omega \end{array}$$

i.e. $f\pi_1 = g\pi_2$, plus a universal property I do not give here.

I present an important property before coming back to Benabou-Mitchel (BM) language. A category $\mathbf{C}$ with finite limits and small Hom-sets (small means to be a set) has a subobject classifier if and only if there is an object Ω and an isomorphism

$$\theta_X : Sub_{\mathbf{C}}(X) \cong Hom_{\mathbf{C}}(X, \Omega)$$

natural for $X \in \mathbf{C}$. It is not important to knowing the definition of natural.

12.6.5 The Example as Such

Now let us come back to BM language. It can be used to describe various objects of $\mathcal{E}$. For example the object of epimorpisms.

A morphism $f : C \to D$ is called an epimorphism if for any object E and any two parallel morphisms, g, h

$$B \rightrightarrows C$$

in $\mathbf{C}$ $gf = hf$ implies $g = h$. One writes $f : C \rightarrowtail D$. One defines, for any two objects X and Y in a topos $\mathcal{E}$, an object $Epi(X, Y) \subseteq Y^X$ called the "object of epimorphisms" from X to Y. This object has the property that $Epi(X, Y) \cong 0$ implies that there is no epimorphism: $X \to Y$.

The BM language can describe various objects. For example, "the object of epimorphisms"

$$Epi(X, Y) \rightarrowtail Y^X$$

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constructed for giving objects X and Y of a topos $\mathcal{E}$, can be described by the expected formulas, involving variables x, y, f of types X, Y, Y^X

$$Epi(X, Y) = \{f \in Y^X | \forall y \in Y \ \exists x \in X \ f(x) = y\}.$$

More explicitly, we (Moerdijk and Mac Lane) state that the subobjects of Y^X defined above in the language of $\mathcal{E}$ coincides with the subobject $Eps(X, Y)$ defined in purely categorical terms.

12.6.6 Some General Remarks

Deriving new valid formulas from given ones can be carried out as for "ordinary", mathematical proofs using variables as if they were ordinary elements, *provided* that the derivation is explicitly constructive. For a general topos, one cannot use indirect proofs (*reductio ad absurdum*) since the law of excluded middle ($\phi \vee \neg\phi$) need *not* be valid, nor can one use the axiom of choice. More technically, this means that the derivation is to follow the rules of the intuitionistic predicate calculus. Kripke's semantics for intuitionistic logic can also be viewed as a description of truth for the language of a suitable topos.

As we saw the existence of a classifier of the functor of subobjects make possible many developments. It is essential to remember that each topos possesses its own logics. That means that the notion of a statement and tools, i.e., logical connectives, are present in any topos. Each topos contains arrows that represent mathematical statements and all logical statements operate on these arrows. The BM language or internal language is a high level language that make possible the manipulation of arrows. The semantics of this internal language is the Kripke-Joyal semantics. One needs to introduce news expressions as a kind of abbreviation for the terms we dispose until now by means of BM language. The introduction of the internal language is a way of giving meaning to the mathematical statements transposed into a topos. The topos becomes in this way a reconstructor of mathematical statements.

12.6.7 Reflexivity

One must more generally consider that the concept of topos, from this point of view, is an in-depth reflection on what a set is. It is, as we have said, a return to oneself of the concept of the whole by extracting from the dynamics that one finds in it a form of self-recovery. But this in-depth reflection on the theory of sets has the consequence of transferring this concept by dynamically recasting it. According to John L. Bell[51] gradually arose the view that the essence of mathematical structure is to

[51] John L. Bell, *Toposes and Local Set Theories*, Clarendon Press, Oxford, 1988, p. 236 sq.

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be sought not in its internal structure as a set-theoretical entity, but rather in the form of its relationship with other structures through the network of morphisms. John Bell says that the uncritical employment of (axiomatic) set theory in their formulation of the concept of mathematical structure prevented Bourbaki from achieving the structuralist objective of treating structures as autonomous forms with no specified substance.

We will pass on this line of analysis from the concept of category to that of topos. Category theory transcends particular structure not by doing away with it, but by taking it as given and generalizing it. And category theory suggests that the interpretation of a mathematical concept may vary with the choice of "category of discourse".[52] And the category theoretic meaning of a mathematical concept is determined only in relation to a "category of discourse" which can *vary*. John Bell states that the effect of casting a mathematical concept in category-theoretic terms is to confer a *degree of ambiguity of reference* on the concept.

It becomes mandatory,[53] to seek a formulation for the set concept that takes into account its underdetermined character, that is, one that does not bind it so tightly to the absolute universe of sets with its rigid hierarchical structure. Category theory furnishes such a formulation through the concept of *topos*, and its formal counterpart *local set theory*. A local set theory is a generalization of the system of classical set theory, within which the construction of a corresponding category of sets can still be carried out, and shown to be a topos. Any topos can be obtained as the category of sets within some local set theory. Topoi are in a natural sense the models or interpretations of local set theories.

12.6.8 Geometric Modalities

Goldblatt, argues (1977)[54] following Lawvere, in favor of a modal interpretation of Grothendieck topology. Modal logic is concerned with the study of one-place connective on sentences that has a variety of meanings, including "it is necessarily the true that", (alethic modality), "it is known that" (epistemic modality), "it is believed that" (doxastic modality), and "it ought to be the case that" (deontic). What we obtained with Grothendieck's topology is what we might call *geometric* modality. Semantically the modal connective corrresponds to an arrow. Lawvere suggests that when the arrow is a topology $j : \Omega \to \Omega$ on a topos, the modal connective has the "natural reading" "it is locally the case that".[55] It is remarkable that the topology becomes thus a way of understanding mathematical reasoning.

[52] John L Bell, *ibid.*, p. 23.

[53] John Bell, *ibid.*

[54] Robert Goldblatt, *Topoi, the categorial analysis of logic*, North-Holland publishing company, Amsterdam New York Oxford, 1977.

[55] Goldblatt, 1977, p. 382.

12.6.9 Some Analogies with the Theory of Relativity

We need to introduce the notion of geometric morphism, that is $\mathbf{E} \rightarrow \mathbf{E}'$. We may think of this morphism as a "nexus between the mathematical worlds represented by $\mathbf{E}$ and $\mathbf{E}'$", or, John Bell adds, "as a method of shifting from $\mathbf{E}$ to $\mathbf{E}'$ and vice versa". There is an analogy here with the physical geometric notion of *change of coordinate system*. In astronomy one effects a change of coordinate system to simplify the description of motions. It also proves possible to simplify the formulation of a mathematical concept by effecting a shift of mathematical framework. Like Bell, we might give as example the topos $\mathbf{Sh}(X)$ of sheaves on X. Here *everything* is varying continuously, so shifting from **Set** to $\mathbf{Sh}(X)$ essentially amounts to placing oneself in a framework which is, according to Bell, so to speak, itself co-moving with the variation over X of any given variable real number. This causes its variation not to be "noticed" in $\mathbf{Sh}(X)$.

Bell notes another analogy. In relativistic physics, invariant physical laws are statements of mathematical physics that, suitably formulated, hold universally, i. e., in every mathematical framework. Analogously, invariant mathematical laws are mathematical assertions that hold universally, i. e., in every mathematical framework. The invariant mathematical laws are those provable *constructively*. Notice in this connection that a theorem of classical logic that is not constructively provable will not hold universally until it has been transformed into its intuitionistic correlate. The procedure of translating classical into intuitionistic logic, Bell said, is thus the logical counterpart of casting a physical law in invariant form.

12.6.10 Brief Complement on Higher Order Logic

We will mention briefly a study that has been made of the relationship between higher-order logic and topoi.[56] Higher order logic[57] has formulae of the form $(\forall X)\phi$ and $(\exists X)\phi$ X may stand for set, a relation, a set of sets, a set of relations, a set of sets of sets..., etc. So for a classical model $\mathfrak{A} = <A, \ldots>$ the range of X may be any of $\mathcal{P}(A)$, $\mathcal{P}(A^n)$, $\mathcal{P}(\mathcal{P}(A^n))$. And as Goldblatt mentions, analogues of these exist in any topos, in the form of Ω^a, $\Omega^{'a}$ etc. and so higher logic is interpretable in $\mathcal{E}$ (a topos). In fact the whole topos becomes a model for a manysorted language, having one sort of individual variables for each $\mathcal{E}$-object. Goldblatt[58] mentions some ancient results by Michael Fourman[59] or by Boileau.[60] This provide a full explication of Lawvere 's

[56] William Goldblatt, *Topoi The categorical analysis of Logic* North–Holland, Amsterdam-New York-Oxford, 1979, p. 286 sq.

[57] We refer not only to second order logics but also to other logics.

[58] Robert Goldblatt, *ibid.* p. 287.

[59] Michael P. Fourman, *Connections between category theory and logic* D. Phil. Thesis Oxford University, 1974.

[60] André Boileau, *Types versus Topos, Thèse de Philosophie Doctor* Université de Montréal, 1975.

statement that “the notion of topos summarizes in objective categorical form the essence of ’higher order logic”’.[61]

12.7 Conclusion

Topos theory involves both geometry, especially sheaf theory, and logic, especially set theory. Nevertheless J. Bell in the book we used for the above remarks, provides a systematic presentation of topos theory from the point of view of formal logic. Does this mean that logic is the discipline that can produce a unification of mathematics? We have seen that logic has in fact transformed itself to become categorical logic. As such, it produces forms of unity and non-unity. The path of this unity is linked to modes of reflection on self reference of mathematics, of which we have shown only certain forms. What is striking is that entire disciplines can be reflected in each other.

We have tried to show that the theory of topos can fulfill this dual unifying and reflective function. We must briefly respond to an objection that might affect our attempt.

When we produce any form of unification, each of the unified disciplines loses much of its substance. And if they are reflected in each other it is in a form that is often very reduced. Hence the claim of “true mathematics” against these theories considered speculative and formal. A factual answer is to say that true mathematics uses these theories more and more precisely because they bring forms of unity and reflection.

It should be added that this unity is not only the result of a formal extraction that “crushes” the information. On the contrary, it is a conceptual element which structures and fills a synthesis. From a Platonic unity, explicitly philosophical, to a topos-unity first in Grothendieck’s work, then in conceptual reflection on it, like that of Lawvere, real mathematics is instead installed on a more synthetic terrain on which they are energized.

The reflexive syntheses proposed by Lautman presented difficulties. He had tried to fill them with an appeal to Heidegger’s philosophy. But this induced other difficulties, notably that of the distinction between dialectics and mathematics. Lautman uses dialectical terms in the Platonic sense and also in the sense of a kind of contradiction theory.

> lf dialectics tries to find its own solutions to the problems it expresses, it will “mimic” mathematics with such a collection of subtle distinctions and logical tricks that it will be mistaken for mathematics itself.[62]

This is the fate of the logicism of Frege and Russell. Nevertheless the line between dialectics and mathematics is neither clear nor stable. And, more difficult for the

[61] William Lawvere, Introduction and ed. for *Toposes, Algebraic Geometry and Logic*, Lectures Notes in Mathematics, Vol. 274, Springer Verlag, 1972.

[62] Lautman (2006, p. 228).

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arguments, mathematics itself can provide dialectical answer to a mathematical question.[63] But this criticism of logicism on behalf of mathematics as such uses the dialectical position, for example, definitions by "abstraction" of equivalence, measurement, operators, etc., characterize not a "genre" in extension but possibilities of structuring, integrations, operations, closure, conceived in a dynamic and organizing way. The distinction within the same structure between the intrinsic properties of a being and its possibilities of action... seems to be similar to the Platonic distinction between the Same and the Other".[64] Using the concept of a topos as a vector of unity and reflexivity, we have made a choice which first has, for him, to have representatives among the important mathematicians of our century. But it appeared to us to express a movement which characterizes mathematics as indefinite movement of the search for this reflexive unity. But this movement of reflexion possesses a mathematical form indissociable from its philosophical questioning. No doubt in a more intrinsic and immanent way than had been proposed by Lautman, who might have opted for this position if he had lived longer. It seems more relevant to search for a mathematical unity of mathematics by means of the concept of a topos, taking into account that this concept possesses philosophical significance. It nevertheless brings us closer to Cavaillès than to Lautman. Lautman in his letter to the mathematician Maurice Fréchet says: "Cavaillès seems to me in what he calls mathematical experience to assign a considerable role to the activity of the mind. There would therefore be no general characters that constitute mathematical reality.... I think in experience there is more than experience."[65] And then he quotes Cavaillès, and we are closer to his position: "Personally I'm reluctant to ask anything else that would dominate the mathematician's actual thinking, I see the requirement in the problems. . . and if dialectics is only that, we only arrive at very general proposition".[66] This has been our position on the issue of the unity reflexivity of mathematics. I can only add that *within* the mathematical experience, philosophical questions arise.

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[63] Brendan Larvor, p. 201.

[64] Albert Lautman, *Les mathématiques les idées et le réel physique*, p. 79 my translation.

[65] A. Lautman, *ibid.* p. 263, my translation.

[66] A. Lautman, *ibid.* p. 263, my translation.

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