Revista Latinoamericana de Etnomatemática
E-ISSN: 2011-5474
revista@etnomatematica.org
Red Latinoamericana de Etnomatemática
Colombia

François, Karen; Vandendriessche, Eric
Reassembling mathematical practices: a philosophical-anthropological approach
Revista Latinoamericana de Etnomatemática, vol. 9, núm. 2, junio-septiembre, 2016, pp. 144-167
Red Latinoamericana de Etnomatemática
San Juan de Pasto, Colombia

François, K., & Vandendriessche, E. (2016). Reassembling mathematical practices: a philosophical-anthropological approach. *Revista Latinoamericana de Etnomatemática*, 9(2), 144-167.

Artículo recibido el 28 de enero de 2016; Aceptado para publicación el 12 de mayo de 2016

Reassembling mathematical practices: a philosophical-anthropological approach[1]

Rassembler des pratiques mathématiques: une approche philosophique et anthropologique

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Abstract

In this paper we first explore how Wittgenstein's philosophy provides a conceptual tools to discuss the possibility of the simultaneous existence of culturally different mathematical practices. We will argue that Wittgenstein's later work will be a fruitful framework to serve as a philosophical background to investigate ethnomathematics (Wittgenstein 1973). We will give an overview of Wittgenstein's later work which is referred to by many researchers in the field of ethnomathematics. The central philosophical investigation concerns Wittgenstein's shift to abandoning the essentialist concept of language and therefore denying the existence of a universal language. Languages—or 'language games' as Wittgenstein calls them—are immersed in a form of life, in a cultural or social formation and are embedded in the totality of communal activities. This gives rise to the idea of rationality as an invention or as a construct that emerges in specific local contexts. In the second part of the paper we introduce, analyse and compare the mathematical aspects of two activities known as string figure-making and sand drawing, to illustrate Wittgenstein's ideas. Based on an ethnomathematical comparative analysis, we will argue that there is evidence of invariant and distinguishing features of a mathematical rationality, as expressed in both string figure-making and sand drawing practices, from one society to another. Finally, we suggest that a philosophical-anthropological approach to mathematical practices may allow us to better understand the interrelations between mathematics and cultures. Philosophical investigations may help the reflection on the possibility of culturally determined ethnomathematics, while an anthropological approach, using ethnographical methods, may afford new materials for the analysis of ethnomathematics and its links to the cultural context. This combined approach will help us to better characterize mathematical practices in both sociological and epistemological terms.

Keywords: Wittgenstein; Language game; Family resemblance; Mathematical practices; String figure-making; Sand drawings; Mathematics and Culture.

Résumé

Dans la première partie de cet article, nous examinons les apports de la philosophie de Wittgenstein à la discussion sur l'existence (simultanée) de pratiques mathématiques différentes, culturellement déterminées.

[1] This article is dedicated to Marcia Ascher (1935-2013) and Paulus Gerdes (1952-2014) we lost recently. They were two of the few founders of the field of ethnomathematics in the 1980s. Their work has deeply influenced our own investigations in this field.

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Nous avançons que les derniers travaux de Wittgenstein offrent un cadre fructueux pouvant servir de fondation philosophique aux ethnomathématiques. De fait, Wittgenstein a été cité dans nombre de travaux - dont nous donnons un aperçu dans cet article - menés dans le champ de l'ethnomathématique. La question philosophique centrale discutée ici concerne l'abandon par Wittgenstein du concept essentialiste de « langage », niant ainsi l'existence d'un langage universel. Les langues - ou « jeux de langage » selon l'expression de Wittgenstein - sont plongé(e)s dans une certaine « forme de vie », dans des formations sociales et culturelles, et dans un ensemble d'activités collectives. Cette idée incite à analyser la rationalité (mathématique) comme une invention - ou une construction - qui a lieu dans des contextes locaux/sociaux spécifiques. Dans une deuxième partie, nous illustrons les idées de Wittgenstein en analysant les aspects mathématiques de deux activités généralement évoquées sous l'appellation « jeux de ficelle » et « dessins de sable », et pratiquées dans diverses sociétés. Une analyse ethnomathématique comparative permet de mettre en évidence des caractéristiques communes et des traits distinctifs dans l'expression d'une rationalité mathématique telle qu'elle s'exprime dans ces deux pratiques, d'une société à l'autre. Enfin, nous suggérons qu'une approche philosophico-anthropologique des pratiques mathématiques est susceptible d'offrir un éclairage nouveau sur les interrelations entre « mathématiques » et « cultures ». Des recherches philosophiques permettent d'avancer sur la question de l'existence des (ethno-)mathématiques culturellement déterminées, alors que dans le même temps l'approche anthropologique - et ethnographique - offre de nouveaux matériaux pour étudier des pratiques à caractère mathématique dans leurs liens avec le contexte culturel. Nous proposons cette double approche dans la perspective de mieux caractériser les pratiques mathématiques dans des termes à la fois sociologiques et épistémologiques.

Mots clés: Wittgenstein; Jeux de langage; Ressemblance familiale; Pratiques mathématiques; Jeux de ficelle; dessins sur le sable; Mathématiques et cultures.

1. INTRODUCTION

A major issue in the field of the philosophy of mathematics is to better understand the extent to which mathematical practices are related to the cultural contexts within which they develop (Larvor 2016). Throughout the last decades, many activities practiced in non-Western cultures (and in societies with an oral tradition in particular) have been identified by ethnomathematicians as being related to mathematics (see Powel & Frankenstein 1997, Gerdes 1999, Ascher 2002). These activities still need further comparison, in an attempt to bring to light invariant and distinguishing features from one cultural context to another, and in order to better characterize mathematical practices (including Western ones) in sociological and epistemological terms. A lot of descriptive and anthropological work is done, and still has to be done, to describe, to analyze and to understand mathematical practices and their relations to culture (Powel & Frankenstein 1997, Gerdes 1999, Ascher 2002). At the same time, one can observe many theoretical investigations in mathematical practices in their relation to culture from the field of philosophy of mathematical practices (Larvor 2016). In order to reassemble mathematical practices and to understand how mathematical practices are connected, we bring together two complementary research directions. We address the issue of the interrelation of mathematics and culture from an interdisciplinary perspective of philosophy and anthropology. An important theoretical investigation of mathematics and culture was carried out by Alan Bishop (1988) with his groundbreaking work on identifying a series of mathematical practices such as counting, locating, measuring, designing, playing as a tool for exploration, and explaining through underlying structures and rules. According to Bishop (1988), these activities can be found in many cultures although he did not ascribe that these mathematical practices were of universal quality. He rather holds that one way of reasoning would be strong and phrased in a particular way in one cultural tradition, and another one in a different culture. Bishop (1988) calls these mathematical practices 'mathematics' with a small 'm' to differentiate from 'Mathematics' with a capital 'M' which stands for the Western body of academic mathematics. Both kinds of mathematics can be seen as a particular answer to a specific question or problem as appearing in a particular environment. Lay-mathematical practices are embedded in specific and local environments while the academic body of Western mathematics is characterized as more universalized and detached from a specific environment.

It was D'Ambrosio (1990) who described mathematical practices in general as human "actions of explaining and understanding in order to survive" (D'Ambrosio 1990, p. 369). He described mathematics as a practice as follows:

> Mathematics is adapted and given a place as "scholarly practical" mathematics which we will call, from now on, "academic mathematics", i.e., the mathematics which is taught and learned in the schools. In contrast to this we will call ethnomathematics the mathematics which is practiced among identifiable cultural groups, such as national-tribal societies, labor groups, children of a certain age bracket, professional classes, and so on. (D'Ambrosio, 1985, p. 45)

Following this research programme we can describe any mathematical practice as culturally identified, be it lay-mathematical practices or the academic body of Western mathematics. Western mathematics is also considered as having developed in a specific historical context, and as continuing to develop, within a particular environment and contextual reality (François & Van Kerkhove, 2010).

Besides these theoretical investigations of the past decades on the relation of mathematical practices and cultures one can observe a new interest in the foundation of the diversity of mathematical practices and their relation to the context they are embedded in. A central question is how we can reassemble these mathematical practices and whether they exist in different cultures. We noticed a similar idea in the work of D'Ambrosio (1985) where he proposes to do a thorough review of the historiography of mathematics, which would take into account all the mathematical knowledge, including non-Western. This project was advanced by many researchers during the past 30 years (see Ascher 1988, 2002; Braunstein 1992, Powel & Frankenstein 1997, Gerdes 1999). For knowledge that has been developed outside Western institutional and scholarly fields, there is the difficulty of tracing the historical development of it ("the chain of historical development") which is, according to D'Ambrosio, "the spine" of "a body of knowledge structured as a discipline": that is why ethnomathematics is not recognized as a structured body of knowledge, but rather as a set of ad hoc practices. D'Ambrosio's (1985) research program goal is thus "to identify within ethnomathematics a structured body of knowledge".

> As things stand now, we are collecting examples and data on the practices of culturally differentiated groups which are identifiable as mathematical practices, hence ethnomathematics, and trying to link these practices into a pattern of reasoning, a mode of thought. Using both cognitive theory and cultural anthropology we hope to trace the origin of these practices. In this way a systematic organization of these practices into a body of knowledge may follow (D'Ambrosio, 1985, p.47)

It is Wittgenstein's philosophy of language that became an interesting philosophy to investigate the interconnection between mathematics, mathematical practices and the local environment from which the practices evolve. Wittgenstein's philosophy (more specific the philosophy of the Philosophical Investigations (PI) from 1939, first –posthumous– published in 1953), provides a conceptual tool to help in the discussion of the possibility of the existence of culturally different mathematical practices. In his Philosophical Investigations, Wittgenstein (1975) abandons the essentialist concept of language and thus denies the existence of a universal language. Languages—or 'language games' (*Sprachspiel*) as Wittgenstein calls them—are immersed in a form of life (*Lebensform*), in a cultural or social formation and are embedded in a totality of communal activities.

Wittgenstein described language as embedded in actual practices within our lives, within our form of life. Language cannot be distinguished from the practice within which a given language gets meaning, as Wittgenstein explains: "Here the term *"language-game"* is meant to bring into prominence the fact that the *speaking* of language is part of an activity, or of a form of life." (Wittgenstein, 1975, §23 p. 127) This idea gives rise to the notion of understanding (mathematical) rationality as an invention or as a construct that emerges in specific local contexts, within a form of life. Based on Wittgenstein's later work and his concept of 'family resemblance' (*Familienähnlichkeit*), philosophers seek to explain different kinds of mathematical knowledge and the coexistence of mathematical practices in different cultures.

2. TOWARDS A PHILOSOPHICAL FOUNDATION OF MATHEMATICAL PRACTICES

Wittgenstein's later work is recently referred to by many researchers in the field of ethnomathematics when looking for a philosophical background in the research field of cultures and mathematics (Ascher, M., & Ascher, R. 1997, Barton 1998, Vilela 2007, Knijnik 2012). At the dawn of ethnomathematical research, we can find a first reference to Wittgenstein in the mathematical and anthropological work of Marcia and Robert Ascher (1997 –first published in 1986). Both researchers (respectively a mathematician and an anthropologist) studied mathematical ideas of people with an oral tradition, based on ethnographic literature. Looking for an understanding of Western mathematics that is relevant to our times and our culture, they argued the importance of relating mathematics and culture as some mathematicians and philosophers did before.

> Some other mathematicians and philosophers, such as Keyser, Kline, Spengler, and Wittgenstein, also realized that mathematics has a cultural context but stopped short of probing other cultures. (Ascher & Ascher, 1997, p. 44)

We will find a second reference to Wittgenstein in the work of Marcia Ascher (1988, 1991) on *Graphs in cultures* and *Trancing Graphs in the Sand*. Ascher (1988, 1991) studied trace figures (with the specification that each line will be traced once and only once without lifting the finger (chalk or pencil) from the ground. These kinds of graphs are found in many different cultures all over the world (e.g. the Kolam Tradition in southern India on which she published). The primary interest of Marcia Ascher lies in discovering and analyzing the mathematical ideas behind these graphs. As an argument to discuss the mathematical nature of these graphs she refers to Wittgenstein's (1956) *Foundations of Mathematics*: "The eminent philosopher, Ludwig Wittgenstein, when trying to define the essence of mathematics, pointed to the figure-tracing problem as one that everyone could recognize as mathematical" (Ascher, 1988, p. 203). Later in 1991, in the work on *Tracing Graphs in the Sand* she refers to the same paragraph: "And the eminent philosopher Ludwig Wittgenstein, in discussing the very foundations of mathematics, used the problem of tracing a quite similar figure as one that captures the essence of the subject" (Ascher, 1991, p. 33). It is clear from this quotation that Ascher refers to Wittgenstein to argue that tracing graphs in the sand can be seen as mathematical figures and thus as mathematical practices. Ascher's first concern at that time was looking for arguments to value some of the indigenous practices as mathematical practices. Therefore, mathematics could no longer be seen as an exclusive 'Western' practice. It was argued that mathematical practices appear in all cultural contexts and that they are related to the local culture, as D'Ambrosio (1990) explained later on. During the First International Conference on Ethnomathematics (ICEm1) which was held at the University of Granada in Spain, we had a new call for a philosophical background of the research field of ethnomathematics. Barton (1998) presented his ideas as follows.

> Ethnomathematicians need to be able to discuss the possibility of the simultaneous existence of culturally different mathematics: it would therefore be helpful to be able to explain how different conceptions of mathematics and different standards of rationality may co-exist. It would be of further help to explain how they can co-exist in cognisance of each other, i.e. how holders of these views may know about the views of others and accept them as right in some rational sense. (Barton, 1998, p. 2)

Barton (1999) published his talk a year later in the International Journal on Mathematics Education: ZDM (*Zentralblatt für Didaktik der Mathematik*) in which he refers explicitly to the Wittgensteinian idea that "we talk mathematics into existence":

> It is suggested that a philosophy based on Wittgenstein may provide ethnomathematics with the position it needs in order to properly describe the objects of mathematics, being the Wittgensteinian idea that we talk mathematics into existence. Shanker's (1987) reading of Wittgenstein proposes that we focus on clarifying what we mean when we talk about mathematics, rather than trying to characterise mathematical

> knowledge. So, rather than arguing about whether mathematical knowledge is certain or fallible, we should recognise that it is created in our talk. Thus mathematics is neither a description of the world, nor a useful science-like theory. It is a system, the statements of which are "rules" for making sense in that system. (Barton, 1999, p. 56)

In his interesting paper on ethnomathematics and philosophy Barton (1999) refers indirectly to the work of Wittgenstein without elaboration on the interrelation of language, meaning and the concepts of mathematics. The first elaborated study on the applicability of Wittgensteinian concepts was carried out by Denise Silva Vilela (2007) who did her PhD thesis (in Portuguese) on the dialogue between mathematical education and the concepts of language-games, family resemblances, life-forms, grammar and rule. Vilela (2010) published the main ideas of her PhD thesis (in English) focussing on the nature of mathematical knowledge.

> […] mathematical knowledge is seen as practice or process rather than a product or a domain of knowledge that may be referenced to 'metaphysical realism' - the notion that an object may be known in itself, in pure form, separate from human practices. […] whereas Wittgenstein does not see mathematics as describing reality; its propositions do not refer to anything that may be discovered, but may be seen as rules or procedures, as models. (Vilela, 2010, p. 346)

Vilela (2010) explains how Wittgenstein's philosophy may plausibly relate to a background philosophy of the ethnomathematics research program. A first fruitful philosophical idea is the non-metaphysical aspect of Wittgenstein's philosophy when he argues that meanings are not fixed or predetermined (Vilela 2010). This is an interesting and moreover, a necessary condition for examining various culturally different mathematical practices. Second, in his Philosophical Investigations Wittgenstein (1975), sought to rid language of referential conceptions in which each word is associated with an object or thing, regardless of human rules or usages. He argued that language must be investigated in the linguistic practice it is situated in. This assertion refers both to the role of language in Wittgenstein's philosophy and to the importance of practices.

> The pursuit is no longer of reality itself, or the form of mental structure identifying a true essence, but the way in which language, as a rule-based system of symbols, shows us the world. Instead of foundations, the focus is how we are inscribed in public language, in the habits or usages of a community, which cannot be explained but only described. If there is a foundation at all, it relates to something that is inseparable from

> linguistic practice: "For what is hidden, for example, is of no interest to us" (Wittgenstein, 1973, PI, § 126) (Vilela, 2010, p. 347).

Vilela (2010) goes back to the initial discussion Ascher (1988) started to explain and to argue that local practices as expressed by local people (e.g. indigenous people, street children or academics) can be seen as a variety of mathematical practices. Since we have no fixed meanings of mathematical concepts nor of mathematical practices, we can understand those local practices as mathematical practices.

> Mathematical practices as used in the street, in schools, in academia, or by professional groups, etc., are a varied set of *language-games* or different uses of mathematical concepts in specific practices and thus do not constitute a single building of knowledge named mathematics, but specific theoretical schemes that shape conditions for the meaning, significance and intelligibility of different situations, times and places in life. (Vilela, 2010, p. 350)

Bases on these philosophical insights Gelsa Knijnik (2012) further investigated the questions on the plurality of mathematics. Knijnik (2012) uses a theoretical toolkit that borrows the concepts of 'language games' and of 'family resemblance' from Wittgenstein (1973, 1975) to analyze mathematical practices from rural life in the south of Brazil. Although Wittgenstein (1973, 1975) uses the expression of 'family resemblance' (*Familienähnlichkeit*) only once in his oeuvre (Wittgenstein, PI, § 67), it became a favourite concept as a useful metaphor to grasp the way how (mathematical) meaning is constructed in a non-essentialist way. The concept 'family resemblance' refers to family members who are characterized by overlapping similarities although no single feature is common to all of them, as shown in the Table 1.

Name	**Eyes**	**Hair**	**Height**	**Physique**
Karen	Green	Red	Tall	Thin
Ubiratan	Blue	Brown	Tall	Thin
Mary	Blue	Red	Short	Thin
Ludwig	Blue	Red	Tall	Fat

Table 1. Representation of the concepts of 'family resemblance' (*Familienähnlichkeit*)

Subsequent Wittgenstein publications (from PI, 1973, 1975) noted that philosophers seek for necessary and sufficient conditions to describe a concept but this craving for generality was misplaced. He refuted the idea that words have a single and fixed meaning by standing for objects in reality (as he explained in his Tractatus). Words acquire meaning from the thoughts of those who are using them in a specific and local context and they are connected by a series of overlapping similarities, as family members are. Wittgenstein explains the expression of 'family resemblance' further by referring to the structure of a thread.

> And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres. (Wittgenstein, 1973, PI, § 67)

To Wittgenstein (1973, PI, § 66) concepts may not have a feature that all its members share but overlapping similarities amongst all members of the concept. The concept he uses to explain this characteristic is the concept of *game* as we referred to in the introduction. The concept of 'game' can differ according to context. No single feature is common to all games.

> And we can go through the many, many other groups of games in the same way; we can see how similarities crop up and disappear. […] And the result of this examination is: we see a complicated network of similarities overlapping and criss-crossing: sometimes overall similarities. (Wittgenstein, 1973, PI, § 66)

The way in which the various uses of the word 'game' are related to one another is the most interesting aspect of language games and an interesting and useful philosophical background to understand the different mathematical practices as they occur in diverse local contexts. The analysis of the logic of mathematical expressions and how they are used can provide us with a deeper understanding of similarities and differences of mathematical practices. In the next section we will present two cases on local practices that resulted from anthropological work from the second author. With these cases we will explore the fruitfulness of Wittgenstein's ideas as referred to in the research field of ethnomathematics.

3. STRING FIGURE-MAKING: HISTORICAL BACKGROUND

String Figure-making consists in applying a succession of operations to a string (knotted into a loop), mostly using the fingers and sometimes the feet, the wrists or the mouth. This succession of operations, generally performed by an individual (sometimes by two individuals working together), is intended to generate a final figure as shown in figure 1[4].

[4] See (Vandendriessche, 2012) and the website "String Figure Algorithms": URL http://www.rehseis.cnrs.fr/www/vandendriessche/StringFigureAlgorithms.html

Figure 1. String figure-making. Left: Bowelogusa, displaying the final figure of *samula kayaula* (a particular river) Oluvilei, Trobriand Islands, Papua New Guinea © Vandendriessche 2007. Right: Josephina, displaying *paloma raity* (the nest of the bird 'paloma'), Santa Teresita, Chaco Paraguay © Vandendriessche 2005

For over a hundred years, string figure-making has been observed by anthropologists in many regions of the world, especially within oral tradition societies. The first scientific study devoted to string figures has been carried out by Cambridge anthropologists Alfred C. Haddon (1855-1940) and William H. R. Rivers (1864-1922). In 1898, they were struck by the prevalence of string-figure-making among the Melanesian islanders in the Torres Strait (South Pacific), so they decided to develop a methodology for recording how each string figure was made (Rivers & Haddon, 1902). Over the 20th century, this methodology (or close ones) has been adopted by other anthropologists to collect (and to publish) the method for making string figures practiced on their own field (Jenness, 1920; Paterson, 1949; Maude 1978…).

Since the end of the 19th century, a few mathematicians have also regarded string figure-making as a worthy topic within their discipline. In particular, Cambridge mathematician Walter William Rouse Ball (1850-1925), a practitioner well acquainted with string figures, perceived the mathematical aspects of this activity and attempted to demonstrate it in a chapter of his book *Mathematical Recreations and Essays* (Ball, 1911, Vandendriessche, 2014a). The analysis of string figure-making practices that the second author studied as an external observer in the Paraguayan Chaco, among the Guarani-Ñandeva[5], and in the Trobriand Islands, Papua New Guinea[6], gives evidence of the expression of a mathematical rationality in this activity (Vandendriessche, 2015a). A string figure-making process can be

[5] This ethnographic research was carried out in 2005 among the Guarani-Ñandeva Indians from the mission Santa Teresita and the village of Laguna negra, both located near the town of Mariscal Estigarribia, Paraguayan Chaco.

[6] The Trobriand Islands are located off the east coast of the main land of Papua New Guinea. This field research was conducted in 2006 and 2007 in the village of Oluvilei on the Island of Kiriwina.

analyzed as a series of ‘simple movements’ that we call ‘elementary operations’ in the sense that the making of any string figure of a given corpus can be described by referring to a certain number of these operations. A String figure can thus be seen as the result of a procedure consisting of a succession of elementary operations as shown in figure 2, 2a and 2b[7].

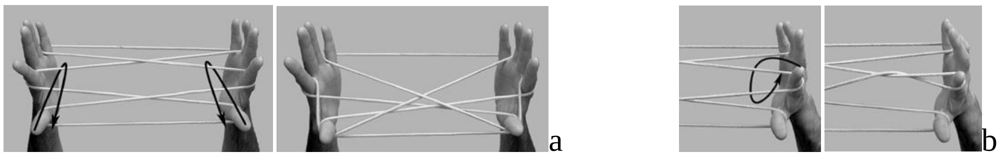


Figure 2. Elementary operations
2a: Operation “picking up” (a string)   2b: Operation “twisting” (a loop)

A ‘sub-procedure’ can then be defined as *any succession of elementary operations either shared, that is, used in the same way in several string figure procedures*, or *iterated in the same one* (iterative sub-procedures). Many sub-procedures[8]—which have a noticeable impact on certain configurations of the string—have been clearly identified, memorized and sometimes named by Guarani-Ñandeva and Trobriander string figure creators/practitioners. Indeed, the concepts of “elementary operation” and “sub-procedure” have been introduced as an observer’s conceptual tools for analytical purposes. However, these concepts sometimes echo through the use of a few vernacular terms referring to movements in string figure-making, suggesting a local perception[9] of the notions of elementary operations and sub-procedures (Vandendriessche, 2014b). Finally, the concept of *transformation* is at work on different levels within both Chaco and Trobriand corpora of string figures. On one hand, this concept is omnipresent since a string figure is the result of the continuous transformation of a loop of string. On the other, analysing the sources suggests that the practitioners worked out how to transform one figure into another. For instance, Trobriand string figure *salibu* (mirror) is transformed in a string figure made with 4 lozenges in a row

[7] See (Vandendriessche, 2015c). URL : http://www.ethnographiques.org/2014/Vandendriessche#3.1
[8] See (Vandendriessche,2015c). URL : http://www.ethnographiques.org/2014/Vandendriessche#3.3
[9] For further discussion on the cultural ideas/practices of the contemporary indigenous people who perform string figures in the societies concerned, see (Vandendriessche, 2012, 2014b, 2015a).

(Figures 3). The latter is also the final figure (modulo a reversal) of the procedure *kalatu gebi navalulu* (linen for young mother)[10].

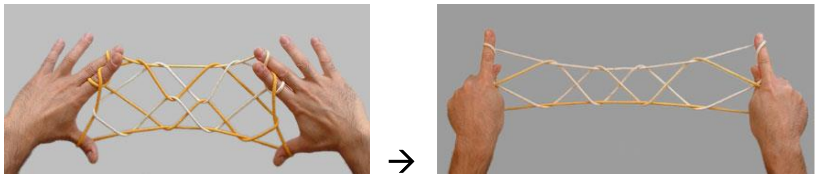


Figure 3. Transformation of the Trobriand string figure *salibu* (mirror) into string figure *kalatu gebi navalulu* (linen for young mother)

The activity of creating new string figure procedures[11] can be regarded as mathematical at different levels. Their production requires an intellectual task of selecting the elementary operations and organizing them in procedures. There is no doubt that this work has consisted in identifying ordered sets of elementary operations—the sub-procedures—having a noticeable impact on different substrata (configurations of the string). String figures thus appear as the result of genuine algorithms. Based on an algorithmic practice, the production of string figure algorithms is also of a "geometrical" and "topological" order, insofar as it is based on investigations into complex spatial configurations, aiming at displaying either a 2-dimensional or a 3-dimensional figure. The transformations of a figure into another, and the iteration of sub-procedures (discussed in the next section), confirm this point.

4. STRING FIGURE-MAKING: ANALYSIS

4.1 Elementary operations and sub-procedures

A comparative study of the "elementary operations" involved in both Chaco and Trobriand corpora of string figures has brought to light some invariant and distinguishing features in the way the string figure algorithms have emerged within these two geographically and culturally distant societies (Vandendriessche, 2015a, pp. 269-283). Apart from a few exceptions, the same elementary operations can be identified in the Trobriand and the

[10] This transformation occurs within the procedure called *mwaya tomdawaya* (name of a person). See URL: http://www.rehseis.cnrs.fr/www/vandendriessche/kaninikula/OpeningsA/OpeningA/59-mwaya-tomdawaya/59-mwaya-tomdawaya.html

[11] In the Trobriand and the Chaco, I (second author) have never met anyone with the ability or even the desire to invent a new string figure. Therefore, we can only speculate about which methods were carried out by the actors to create new string figure algorithms.

Chaco corpora. However, a statistical analysis has showed that the two corpora differ in the occurrences of certain elementary operations, as well as, in the fingers that implement them. For instance, (see Figure 4) the frequency of the operation "hooking down (a string)" (Figure 4a) is 5% of the total number of elementary operations involved within the Chaco corpus, whereas its frequency is only 1% within the Trobriand corpus. Furthermore, the operation "enlarging" (Figure 4b) does not occur in the latter, whereas it is used seven times in the Chaco corpus.

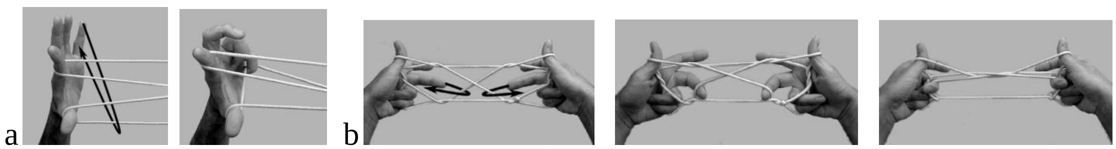


Figure 4. Two elementary operations
4a. Operation "hooking down" (a string) (1-2) 4b. Operation "enlarging" (a hole) (3-5)

Finally, the operation "picking up" is implemented with the thumbs (as in Figure 2) (resp. with middle fingers) in 44.3% of the cases (resp. 2.6%) in the Trobriand corpus and 24.4% (resp. 20.7%) in the Chaco corpus. Although it is still difficult to accurately describe the phenomenon, one can hypothesize that the latter variations have had a significant impact on the creation of string figure procedures, and hence on the shape/structure of the string figures corpus from one society to another.

A few sub-procedures, all consisting in a small number of elementary operations, can be found in the making of both Guarani-Ñandeva and Trobriander string figures. For instance, the sub-procedure "transferring (a loop)" (Figure 5)—consisting in two elementary operations—occurs frequently in both corpora. By contrast, all sub-procedures consisting in a greater number (more than 3) of elementary operations are different from one corpus to another. Unlike the elementary operations themselves, the combination of these elementary operations into sub-procedures appears to be a clear distinguishing feature from one corpus to another[12] (Vandendriessche, 2015a).

[12] See (Vandendriessche, 2015c). URL: http://www.ethnographiques.org/2014/Vandendriessche#3.3

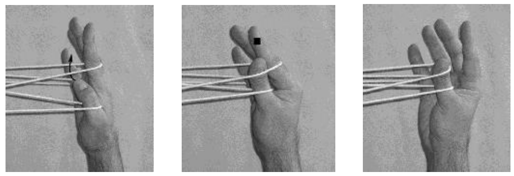


Figure 5. Two elementary operations: "Transferring" = "Inserting" + "releasing"

4.2 The concept of iteration

The concept of iteration is involved in the "iterative sub-procedures" previously defined as sub-procedures [which are] iterated several times within a given string figure algorithm. Sometimes, the creators of string figures worked out these singular sub-procedures to display a particular pattern (Figure 6) as part of the final figure and as many times as the number of iterations[13].

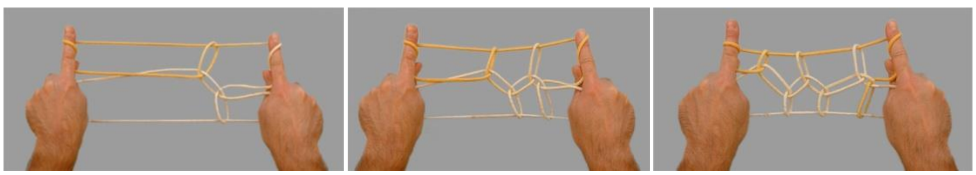


Figure 6. Iteration of a pattern: Trobriander string figure procedure *budi-budi* (name of an island)

Iterative sub-procedures rarely occur in the Chaco corpus, whereas the iteration of a sub-procedure is omnipresent in the Trobriand corpus. More generally, such iterative sub-procedures can be often found in the Austronesian string figure corpora (Jenness, 1920; Maude 1978; Noble, 1979). By contrast, one can notice the rareness of iterative sub-procedures in South America corpora (Martinez-Crovetto, 1970; Braunstein, 1992; Sturzenegger, 1992). This seems to indicate that the principle of iteration should be an efficient conceptual tool to differentiate string figure corpora.

4.3 "Topological" comparison

The "heart-sequence" string figure mathematical tool, created by the Navaho American mathematician Thomas Storer (1938-2006) is an efficient conceptual tool to better understand the impact of ordered sets of elementary operations on particular configurations of the string (Storer, 1988). Passing the string around a finger forms a "loop". Storer pointed out that many string figures all over the world can be seen as the result of

[13] See (Vandendriessche, 2015c). URL: http://www.ethnographiques.org/2014/Vandendriessche#3.4

sequences of operations implemented on the “loops”, such as the insertion of a loop into another. In other words, if one had the opportunity to perform a string figure in the dark with a fluorescent string, the movements of the string could be summarized in a certain number of such operations on the loops. By focusing on these movements during the process, and by converting them into a mathematical formula, the heart-sequence gives—what I call—a “topological” view of a string figure algorithm (Vandendriessche, 2015b).

The concept of heart-sequence enables us to shed new light on certain phenomena which frequently occur in the string figure corpora, such as the transformation of one figure into another and the making of particular patterns (Vandendriessche, 2015a). Moreover, this conceptual tool enables me to bring to light *cultural specificities*.

The comparative analysis of certain sub-procedures—through their heart-sequences—suggests that Guarani-Ñandeva and Trobriander practitioners certainly considered the configuration of loops in different ways. In the Chaco, the insertion of a loop into another is often based on the insertion of a loop created on one hand through an opposite loop carried by the other hand. This phenomenon is quite rare in the Trobriands corpus, where the insertions mostly involve the loops carried by the same hand[14].

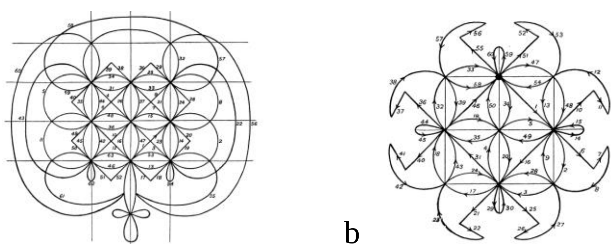
The latter outcomes bring to light that besides an apparent uniformity one can detect/reveal cultural differences in the practices of string figure-making from one society to another. The next section will give another example of such cultural variations regarding another (ethno-)mathematical activity.

5. SAND DRAWING: HISTORICAL BACKGROUND

As well as string figures, figure-making through drawing a continuous line with the finger, either in the sand or on dusty ground, without retracing any part of the drawing, has been observed in various societies. Since the early 20th century, a number of ethnographers have published collections of sand drawings that also include remarks about the cultural contexts

[14] The Trobriander String figure *misima* (name of an island) and the Guarani-Nandeva string figure “Pala - Huella de wanako - Ovecha ija” (shovel-trail of the ‘wanako’-trail of the goat) illustrate these two different heart-sequence schemes: http://www.rehseis.cnrs.fr/www/vandendriessche/kaninikula/OpeningsA/OpeningA/sg-misima/44.Misima/44.Misima.html
http://www.rehseis.cnrs.fr/www/ vandendriessche/tukumbu/openingsP/P1/sg-pala/38-seriesIV/38-seriesIV.html

of their production. Working from these sources, ethnomathematicians have highlighted the algorithmic aspects of the practice of sand drawing, while bringing to light significant variations from one society to another. In the 1920s, anthropologist Bernard Deacon collected 91 sand drawings in Malekula Island in the New Hebrides (Vanuatu since 1980), South Pacific, where these drawings are called *nitus*, and generally drawn through the framework of a grid made of perpendicular lines as illustrated in Figure 7 (7a and 7b) (Deacon & Wedgwood, 1934). Most of these sand drawings are "mono-linear" —*i.e.* made with one and only one continuous line. However, a few of these patterns are "poly-linear", which means that they are made with several interlaced lines. All of these drawings are symmetrical.



a b

Figure 7. Malekula Sand drawings (Deacon & Wedgwood, 1934)
7a. *levwaa* (the banana stump) 7b. *nimingge* (variety of yam)

6. SAND DRAWING: ANALYSIS

In the article "Graphs in cultures" (1988) and later in her 1991 book "Ethnomathematics: A multicultural view of mathematical ideas", ethnomathematician Marcia Ascher demonstrates that some drawings collected by Deacon can be described through an 'algebra of processes'. Ascher introduces the term algebra in the sense that while making such a drawing one is dealing with geometrical patterns obtained through 'tracing procedures' that she defines as 'entities'. One can operate on these entities by applying 'precise rules' defined as a set of 'processes' or 'basic transformations'. Ascher defines a "process" as the way in which a tracing procedure is modified. If, beginning from the end of tracing procedure A, the same procedure is repeated, the process is identity, the resulting procedure is still A, and the overall procedure is A followed by A, which Ascher symbolizes by AA. If, however, when drawing the second segment every motion in A is rotated clockwise through 90°, the process is 90° rotation, the new procedure is identified symbolically as

A_{90}, and the overall procedure is AA_{90} (as illustrated in Figure 8). In all, the processes of this algebra are: identity, reflection over either a vertical (V) or a horizontal (H) line; rotation through 90°, 180°, or 270°; and each of these can be simultaneous with inversion, which reverses the order of the drawing procedure (noted "$\bar{A}$" by Ascher). There are, therefore, twelve such processes (Ascher, 1991, pp. 51-52).

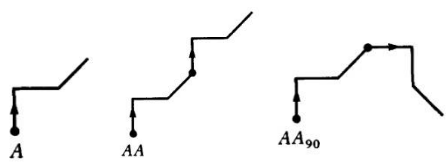
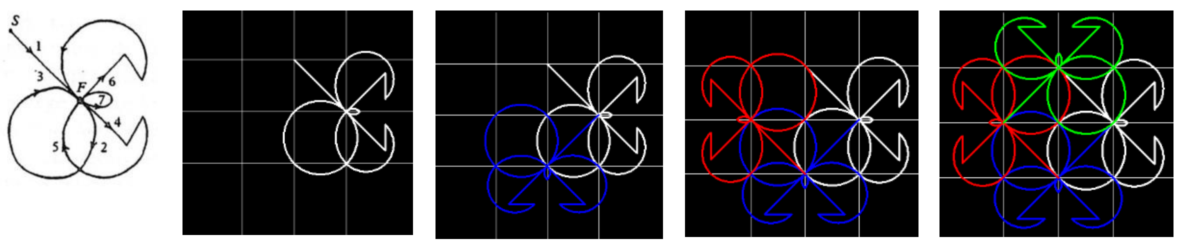


Figure 8. A few of the processes (extracted from Ascher 1991)

The sand drawing "yam" (Figure 7b) collected by Deacon & Wedgwood can be analyzed in terms of "tracing procedures" and "processes" specific to it: when the initial procedure A is selected as the basic tracing unit, it becomes possible to concisely describe the overall tracing procedure as $AA_{90}A_{180}A_{270}$ as shown in Figure 9.



A $\quad A_{90} \quad A_{180} \quad A_{270}$

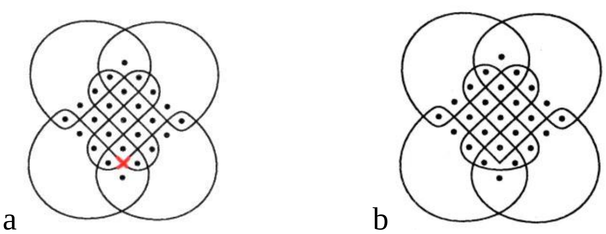
Figure 9. Tracing procedure and processes for the making of the sand drawing "yam"

The sand drawing "yam"—as well as many other Malekula sand drawings collected by Deacon—can thus be formalized as an ordered sequence of "basic transformations" of a given "tracing procedure of a basic pattern". The work in progress that I (second author) am currently carrying out on sand drawings made by the Islanders of Ambrym, Vanuatu, confirms Ascher's analysis[15]. Furthermore, the existence of technical vernacular terms used

[15] https://www.mpiwg-berlin.mpg.de/en/research/projects/departmentSchaefer_WG_AofJ_StringFiguresAndSandDrawings

by the sand drawing contemporary practitioners to designate “basic patterns” or “movements” seems to indicate a local perception of Ascher’s formalization.

A similar practice of sand drawing is called *sona* by the Tchokwe, a cultural group from North-East Angola, Zaire and Zambia. A large number of Tchokwe sand drawings were collected and published by anthropologist Mario Fontinha in the 1980s (Fontinha, 1983). In his 1995 book, ethnomathematician Paulus Gerdes demonstrated that 80 percent of *sona* published by Fontinha are symmetrical. 75 percent have at least one axis of symmetry, and often a double symmetry. Invariant *sona* through rotational symmetry are less common. Moreover, 61 percent of sand drawings are mono-linear. These statistical data allowed Gerdes to conclude that the creators of these procedures preferred mono-linear drawings with a certain degree of symmetry. More precisely, the sources suggest that these creators have sometimes investigated sand drawing procedures dealing with both concepts of symmetry and mono-linearity simultaneously. Gerdes gives the example of two very similar sand drawings (which carry the same vernacular name), differing in one and only one crossing. These two drawings are most likely variation of one another (see Figure 10). The bi-linear drawing A (Figure 10a), with two (vertical and horizontal) axes of symmetry, could have been transformed into the mono-linear drawing B (Figure 10b) by deleting an intersection point of the two components of the drawing A, and consequently losing the double symmetry.



a b

Figure 10. Variation on the sand drawing *sako rya uyanga* (little knots that symbolize the tails of certain animals). By deleting the red crossing (my emphasis) the bi-linear *lusona* (8a) with a double symmetry is transformed into a mono-linear *lusona* (8b) with a single axis of symmetry.

The concepts of symmetry and mono-linearity are then omnipresent within the Tchokwe corpus, as well as in the Vanuatu corpus of sand drawings. However, besides this common feature, Gerdes’ analysis of Tchokwe sand drawings suggests significant differences in the cognitive acts underlying the creation of these patterns in these two societies. First, *sona* are made around a grid of equally spaced dots. Even though one can theoretically transform such a grid into a grid made of parallel lines, as the ones involved in the Vanuatu corpus, it is very likely that these different frameworks led the practitioners in different directions, creating very different basic patterns. Secondly, putting apart a few exceptions, equally spaced dots led Tchokwe practitioners to create drawings that can be seen as 4-regular graphs[16] (cf. Figure 9), whereas the graphs extracted from the *nitus* are generally not regular. Finally, Gerdes demonstrated that a large number of *sona* have been created using "plaited rectangular patterns" and some 'rules' that have allowed the Tchokwe practionners to create new sand drawings as illustrated in Figure 11 (11a and 11b). In particular, five of these rules, called "sequencing rules" by Gerdes, allow these practitioners to concatenate these 'plaited' patterns in order to create more sophisticated sand drawings (Gerdes, 1995).

Figure 11. Concatenating plaited patterns.
11a. Rectangular plaited patterns 11b. Drawing *lambo rya kajama* (leopard skin) resulting from the second sequencing rule (extracted from Gerdes, 1995, p. 139 & 169)

For instance, the second sequencing rule states that a new mono-linear pattern is created when two mono-linear patterns meet tangentially. The use of this sequencing rule very likely underlie the creation process of the sand drawing *lambo rya kajama* (leopard skin) shown in Figure 11b.

An additional rule (previously mentioned – cf. Figure 10) is called "deletion rule" by Gerdes: precisely, it states that if one deletes an intersection point of a mono-linear "plaited rectangular" pattern, then the latter becomes bi-linear. Moreover, by deleting an intersection point of the two resulting components, the latter bi-linear pattern is transformed into a mono-linear one. The combination of this rule with the second sequencing rule has certainly led Tchokwe practitioners to create new sand drawings through a process that

[16] A 4-regular graph is a graph whose the degree of each vertex is equal to 4 *i.e.* there are 4 edges incident to the each vertex.

Gerdes was able to reconstruct, as illustrated by the reconstruction of the sand drawing in Figure 12, 12a, 12b, 12c and 12d.

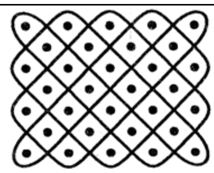
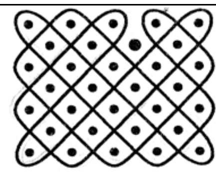
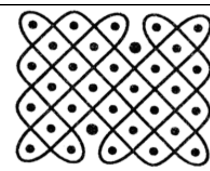
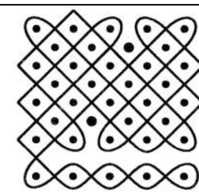
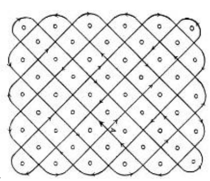
			
12a. mono-linear pattern	12b. bi-linear pattern	12c. mono-linear pattern	12d. Sand drawing "hyena" which represents a hyena catching a goat

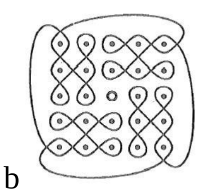
Figure 12. Reconstruction of sand drawing making process (Gerdes, 1995, p. 167-168).

There is evidence that comparable/similar investigations have been sometimes carried out by the Tchokwe and Ni-Vanuatu creators of sand drawings. For instance, the Ni-Vanuatu sand drawing *Nevses* is based on a mono-linear "plaited rectangular pattern"; and the Tchokwe sand drawing *usake wa bundu* can be analyzed as the iteration of a particular motif under a succession of 90° rotations as in the Malekula sand drawing "Yam" (Figure 13, 13a and 13b). However, the comparison of Ascher and Gerdes' analysis of Malekula and Tchokwe sand drawing corpora suggests that a significant number of sand drawings in each corpus have been created through very different methods—or operational schemes—from one society to the other.



a

13a- Malekula Sand drawing *nevses* (meaning unkown) (Deacon & Wedgwood, 1934, p. 175)



b

13b- Tchokwe sand drawing *vandumba zia vantu* (human-lion)(Gerdes, 1995 p. 59)

Figure 13. Similar investigations from Angola to Vanuatu

7. DISCUSSION

In the first section of this paper we gave an overview of a possible and fruitful philosophical background of the ethnomathematics research program based on the late philosophy of Wittgenstein. By doing so, we will further investigate a philosophical interpretation of mathematical practices to bring new arguments for the understanding of mathematics as a cultural product. We use the philosophical background as a theoretical framework to better understand anthropological work on mathematical practices and the value anthropologists give to local mathematical practices. As shown in this article, Wittgenstein's concept of 'game' is seen as a fruitful concept by many ethnomathematicians, to understand the culturally different mathematical practices. The specificity of this concept is that it has no single feature that is common to all games, although the concept includes the inherent relations. We explained this special feature with the example of a family where no individual has all features common to all family members, although all members of one family are related in one way or another. Wittgenstein used the notion of family resemblances to explain the way meaning is constituted in relation to the use of it. Looking at the 'language games' of mathematical practices, a central question is how we identify and recognize other ways of mathematical reasoning, other mathematical language games or other ethnomathematics. This analysis is the core research topic of the anthropologists studying (indigenous) mathematical practices in order to analyse those forms of rationality that underlie different mathematical practices. In the second section of this paper we introduced, analysed and compared the mathematical aspects of two activities known as string figure-making and sand drawing. Each of these activities is practiced—at first sight—in comparable/similar ways in various oral tradition societies around the world, and can be analysed as mathematical practices. However, in both cases, an (ethno-)mathematical/ethnomodeling comparative analysis has given evidence of either formal similarities or differences in the ways the activity of string figure-making (resp. "sand drawing") is practiced, on the one hand, by the Trobrianders from Papua New Guinea, and, on the other, by the Guarani-Ñandeva from Chaco, Paraguay (resp. Tchokwe, Angola and Malekula, Vanuatu) . Working in this way, we have brought to light some invariant and distinguishing features—from one society to another—in the expression of the mathematical rationality that underlie the creation of these artefacts. A local perception of the procedural aspects of both string figure-making and sand drawing practices has been argued in the present paper. However, the latter has mostly followed an analytical approach (etic perspective), aiming to make hypotheses about the ways practitioners have created and explored these procedures (in the past). It will be indeed of fundamental importance to question further these results through an 'emic' perspective, in the light of contemporary practitioners' own conceptualization of these practices.

In this paper we suggest that a philosophical-anthropological approach of mathematical practices like string figure-making and sand drawing may allow us to better understand the interrelations between mathematics and cultures. Philosophical investigations may help the reflection on the possibility of culturally determined ethnomathematics, while an anthropological approach, using ethnographical methods, may afford new materials for the analysis of ethnomathematics and its links to the cultural context. This combined approach should help us to better characterize mathematical practices in both sociological and epistemological terms.

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