Parthood and Size in Euclid (and Beyond)

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1. The first part of my talk is concerned with Euclid's attempt in Book I and II of his ELEMENTS to base the "megethology" of "rectilinear figures" (polygons) upon their mereology. One of his main problems is to find out whether two such figures are the same size or one is smaller than the other. The general problem is reduced by him to the special case of triangles. Furthermore, he shows that the size of triangles depends in a complex way upon the size of their sides and angles. Thus he finally reaches at the task of comparing line segments and angles as a basis for the size comparison of triangles.

There are two cases where the size relation between two items (line segments, angles, or triangles) is obvious, namely: (1) if one is a proper part of the other and (2) when they coincide. In the first case the part is smaller than the whole, in the latter they are of equal size. Mereological relations thus provide a direct clue to relationships of size in these two distinguished cases. Euclid tries to reduce the more general case that the items to be compared are disjoint ("at a distance from each other") to the distinguished simple cases by using the principle

$$x \text{ is smaller than } y \text{ if } x \text{ is the same size as a proper part } z \text{ of } y. \qquad \text{(S)}$$

Thus, in order to find out whether x is smaller than y we have to "move" x upon y. "Moving" x upon y (Euclid's ἐφαρμόζειν, 'apply', 'fit'; cf. I.4, I.8, III.24) I interpret as constructing (by circle and straightedge) a "copy" of x (which thus actually "remains at its place") upon y. If the copy of x covers only a part of y, then x is smaller than y. If, however, it coincides with y, x and y are equal size. Finally, if y extends over the copy, it is larger than x.

This strategy is carried out by Euclid for the case of line segments in the first two theorems of Book I. The case for angles, however, is more complicated. I.23 might be used to "move" an angle upon another one. However this theorem depends upon I.4 and I.8 which concern the equality (congruence) of triangles and are themselves proved by the method of superposition. Thus I.23 already presupposes that one is able to compare triangles as regards their size and thus cannot be used in the construction of the "covering" triangles in the proofs of I.4 and I.8. Hence an attempt to reduce the comparisons of triangles to that of their sides and angles seem to lead up to a circle. I shall consider this situation in more detail in my talk.

2. The second part of my talk is devoted to the fate of Euclid's attempt to base the megethology of rectilinear figures upon their mereology. Given Euclid's impact upon the development of mathematics, it does not come to a big surprise that his principle (S) surfaces in the writings of many later authors. Two examples may suffice here: Leibniz's INITIA RERUM MATHEMATICARUM METAPHYSICA [5, p. 20] and Bolzano's WISSENSCHAFTSLEHRE [1, §97].

Euclid's approach to the problem of "comparing magnitudes" is still implicit more than 2000 years later in Otto Hölder's seminal article [2] from 1901 on the theory of measurement — an article which has been called "a watershed in measurement theory, dividing the *classical* (stretching from Euclid) and the modern [...] eras," [4, p. 235]. Hölder aknowledges his debts to Euclid when stating that "[t]he theory of measurable magnitudes was developed to a high level by Euclid" [4, p. 238] = [2, p. 3]. In 1924 he considers the topic of measurement from a more philosophical and foundational rather than from a technical perspective in the third chapter of his extensive monograph about THE MATHEMATICAL METHOD. Using the measurement of line segments as an example case, he (p. 57) adopts (S) in the following guise: "In a wider sense of the word [*part*], the line segment $A''B''$ may be called a part of the line segment AB if $A''B''$ is equal in size to a proper part $A'B'$ of AB." He hurries to add then: "However, I would like to avoid the term *part* in that case and would prefer to say that AB is larger than $A''B''$." The notion of part in this explanation of the larger-than-relationship is defined by the between-relation connecting the points of a line segment. $A'B'$ is a part of AB (all points incident with the same segment) iff either $A' \neq B'$ and both A' and B' are between A and B or only one of A' and B' is between A and B and the other point coincides either with A or with B.

For Hölder the concept of part is what he calls a "synthetic concept", i.e., a concept which is constructively defined on the base of concepts which are simply accepted as given ("gegeben"). In the case of the parts of a line segment the "given" concept is the topological relation of betweeness. This relation, as we have just seen, can be used to define the part-whole-relation between line segments (cf. Hölder's definition in the previous paragraph). Topological notions, common to us today, are missing from Euclid's conceptual framework. However, in her study of the role of mathematics in Kant's critical philosophy, Lisa Shabel [6, p. 17] observes that "[t]he definitions of Book I of Euclid's ELMENTS present a topological picture of two-dimensional Euclidean space wherein part/whole relations (or, more generally, a principle of spatial containment) emerges as fundamental." Indeed, using Euclid's framework, beweeness is readily definable: B is between A and C if the line segment AB is a part of AC. Thus it seems that one of the things which happened at the watershed marked by Hölder's article is a reversal of the views of what is "given" and what results from a constructive "synthesis": part-whole relationships formerly considered to be "given" are now — if recgnized at all — conceived as being defined in terms of topological notions.

References

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[4] Hölder, Otto: "The Axioms of Quantity and the Theory of Measurement." English translation of Part I of [3] with an introduction by Joel Michell. *Journal of Mathematical Psychology* 40. 235–252.

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