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A SOURCE BOOK
in
MATHEMATICS

By
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SOURCE BOOKS IN THE HISTORY OF THE SCIENCES

General Editor's Preface

THIS series of Source Books aims to present the most significant passages from the works of the most important contributors to the major sciences during the last three or four centuries. So much material has accumulated that a demand for selected sources has arisen in several fields. Source books in philosophy have been in use for nearly a quarter of a century, and history, economics, ethics, and sociology utilize carefully selected source material. Recently, too, such works have appeared in the fields of psychology and eugenics. It is the purpose of this series, therefore, to deal in a similar way with the leading physical and biological sciences.

The general plan is for each volume to present a treatment of a particular science with as much finality of scholarship as possible from the Renaissance to the end of the nineteenth century. In all, it is expected that the series will consist of eight or ten volumes, which will appear as rapidly as may be consistent with sound scholarship.

In June, 1924, the General Editor began to organize the following Advisory Board:

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* No longer chairman of a committee, because of the pressure of other duties, but remains on the Board in an advisory capacity.

Each of the scientists on this board, in addition to acting in a general advisory capacity, is chairman of a committee of four or five men, whose business it is to make a survey of their special field and to determine the number of volumes required and the contents of each volume.

In December, 1925, the General Editor presented the project to the Eastern Division of the American Philosophical Association. After some discussion by the Executive Committee, it was approved and the philosophers of the board, with the General Editor as chairman, were appointed a committee to have charge of it. In November, 1927, the Carnegie Corporation of New York granted $10,000 to the American Philosophical Association as a revolving fund to help finance the series. In December, 1927, the American Association for the Advancement of Science approved the project, and appointed the General Editor and Professors Edwin G. Conklin and Harlow Shapley a committee to represent that Association in cooperation with the Advisory Board. In February, 1928, the History of Science Society officially endorsed the enterprise. Endorsements have also been given by the American Anthropological Association, the Mathematical Association of America, the American Mathematical Society, and the American Astronomical Society within their respective fields.

The General Editor wishes to thank the members of the Advisory Board for their assistance in launching this undertaking; Dr. J. McKeen Cattell for helpful advice in the early days of the project and later; Dr. William S. Learned for many valuable suggestions; the several societies and associations that have given their endorsements; and the Carnegie Corporation for the necessary initial financial assistance.

Gregory D. Walcott.

Long Island University,
Brooklyn, N. Y.
December, 1928.

A SOURCE BOOK IN MATHEMATICS

Author's Preface

The purpose of a source book is to supply teachers and students with a selection of excerpts from the works of the makers of the subject considered. The purpose of supplying such excerpts is to stimulate the study of the various branches of this subject—in the present case, the subject of mathematics. By knowing the beginnings of these branches, the reader is encouraged to follow the growth of the science, to see how it has developed, to appreciate more clearly its present status, and thus to see its future possibilities.

It need hardly be said that the preparation of a source book has many difficulties. In this particular case, one of these lies in the fact that the general plan allows for no sources before the advent of printing or after the close of the nineteenth century. On the one hand, this eliminates most of mathematics before the invention of the calculus and modern geometry; while on the other hand, it excludes all recent activities in this field. The latter fact is not of great consequence for the large majority of readers, but the former is more serious for all who seek the sources of elementary mathematics. It is to be hoped that the success of the series will permit of a volume devoted to this important phase of the development of the science.

In the selection of material in the four and a half centuries closing with the year 1900, it is desirable to touch upon a wide range of interests. In no other way can any source book be made to meet the needs, the interests, and the tastes of a wide range of readers. To make selections from the field, however, is to neglect many more sources than can possibly be selected. It would be an easy thing for anyone to name a hundred excerpts that he would wish to see, and to eliminate selections in which he has no special interest. Some may naturally seek for more light on our symbols, but Professor Cajori's recent work furnishes this with a satisfactory approach to completeness. Others may wish for a worthy treatment of algebraic equations, but Matthiessen's *Grundzüge* contains such a wealth of material as to render the undertaking unnecessary. The extensive field of number theory will appeal to many readers, but the monumental work of Professor Dickson, while not a source book in the ordinary sense of the term, satisfies most of the needs in this respect. Consideration must always be given to the demands of readers, and naturally these demands change as the literature of the history of mathematics becomes more extensive. Furthermore, the possibility of finding source material that is stated succinctly enough for purposes of quotation has to be considered, and also that of finding material that is not so ultra-technical as to serve no useful purpose for any considerable number of readers. Such are a few of the many difficulties which will naturally occur to everyone and which will explain some of the reasons which compel all source books to be matters of legitimate compromise.

Although no single department of "the science venerable" can or should be distinct from any other, and although the general trend is strongly in the direction of unity of both purpose and method, it will still serve to assist the reader if his attention is called to the rough classification set forth in the Contents.

The selections in the field of Number vary in content from the first steps in printed arithmetic, through the development of a few selected number systems, to the early phases of number theory. It seems proper, also, to consider the mechanics of computation in the early stages of the subject, extending the topic to include even as late a theory as nomography. There remains, of course, a large field that is untouched, but this is a necessary condition in each branch.

The field of Algebra is arbitrarily bounded. Part of the articles classified under Number might have been included here, but such questions of classification are of little moment in a work of this nature. In general the articles relate to equations, symbolism, and series, and include such topics as imaginary roots, the early methods of solving the cubic and biquadratic algebraic equations and numerical equations of higher degree, and the Fundamental Theorem of Algebra. Trigonometry, which is partly algebraic, has been considered briefly under Geometry. Probability, which is even more algebraic, is treated by itself, and is given somewhat more space than would have been allowed were it not for the present interest in the subject in connection with statistics.

The field of Geometry is naturally concerned chiefly with the rise of the modern branches. The amount of available material is such that in some cases merely a single important theorem or statement of purpose has been all that could be included. The topics range from the contributions of such sixteenth-century writers as Fermat, Desargues, Pascal, and Descartes, to a few of those who, in the nineteenth century, revived the study of the subject and developed various forms of modern geometry.

The majority of the selections thus far mentioned have been as non-technical as possible. In the field of Probability, however, it has been found necessary to take a step beyond the elementary bounds if the selections are to serve the purposes of those who have a special interest in the subject.

The fields of the Calculus, Function Theory, Quaternions, and the general range of Mathematics belong to a region so extensive as to permit of relatively limited attention. It is essential that certain early sources of the Calculus should be considered, and that some attention should be given to such important advances as relate to the commutative law in Quaternions and Ausdehnungslehre, but most readers in such special branches as are now the subject of research in our universities will have at hand the material relating to the origins of their particular subjects. The limits of this work would not, in any case, permit of an extensive offering of extracts from such sources.

It should be stated that all the translations in this work have been contributed without other reward than the satisfaction of assisting students and teachers in knowing the sources of certain phases of mathematics. Like the editor and the advisory committee, those who have prepared the articles have given their services gratuitously. Special mention should, however, be made of the unusual interest taken by a few who have devoted much time to assisting the editor and committee in the somewhat difficult labor of securing and assembling the material. Those to whom they are particularly indebted for assistance beyond the preparation of special articles are Professor Lao G. Simons, head of the department of mathematics in Hunter College, Professor Jekuthiel Ginsburg, of the Yeshiva College, Professor Vera Sanford of Western Reserve University, and Professor Helen M.

Walker, of Teachers College, Columbia University. To Professor Sanford special thanks are due for her generous sacrifice of time and effort in the reading of the proofs during the editor's prolonged absence abroad.

The advisory committee, consisting of Professors Raymond Clare Archibald of Brown University, Professor Florian Cajori of the University of California, and Professor Leonard Eugene Dickson of the University of Chicago, have all contributed of their time and knowledge in the selection of topics and in the securing of competent translators. Without their aid the labor of preparing this work would have been too great a burden to have been assumed by the editor.

In the text and the accompanying notes, the remarks of the translators, elucidating the text or supplying historical notes of value to the reader, are inclosed in brackets [ ]. To these contributors, also, are due slight variations in symbolism and in the spelling of proper names, it being felt that they should give the final decision in such relatively unimportant matters.

DAVID EUGENE SMITH.

NEW YORK,
September, 1929.

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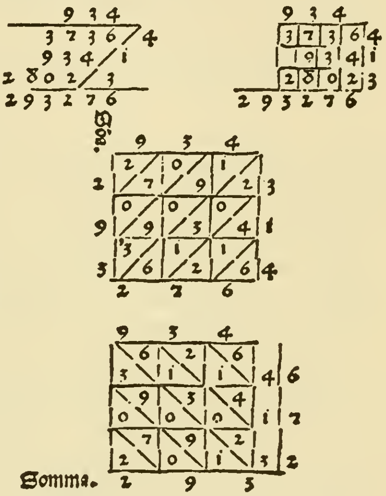
although the results are the same. Now not to speak at too great length I say in brief, but sufficiently for the purposes of a Practica, that there are three methods of multiplication, viz., by the tables, cross multiplication, and the chess-board plan. These three methods I will explain to you as briefly as I am able. But before I give you a rule or any method, it is necessary that you commit to memory the following statements, without which no one can understand all of this operation of multiplication[1]. . .

I have now given you to learn by heart all the statements needed in the Practica of arithmetic, without which no one is able to master the Art. We should not complain, however, at having to learn these things by heart in order to acquire readiness; for I assure you that these things which I have set forth are necessary to any one who would be proficient in this art, and no one can get along with less. Those facts which are to be learned besides these are valuable, but they are not necessary.

Uoglio pero che tu intendi che ſono altri modi de molnplicare per ſcachiero:li quali laſſaro al ſtudi o tuo:mettendo li exempli ſoi ſolamente in forma. come porai vedere qui ſotto

Or togli de fare lo predito ſcachiero.zoe.3 i 4. fia.9 3 4.e nota de farlo per li quatro modi come qui va ſotto.



Having learned by heart all of the above facts, the pupil may with zeal begin to multiply by the table. This operation arises when the multiplier is a simple number, and the number multiplied has at least two figures, but as many more as we wish. And that we may more easily understand this operation we shall call the first figure toward the right, units; the second toward the left, tens, and the third shall be called hundreds. This being understood, attend to the rule of working by the table, which is as follows: First multiply together the units of the multiplier and

[1] [The author now gives the multiplication table, omitting all duplications like 3 × 2 after 2 × 3 has been given, but extending for "those who are of scholarly tastes" the table to include multiples of 12, 20, 24, 32 and 36, as needed in the monetary systems used by merchants of the time.]

KUMMER

On Ideal Numbers

(Translated from the German by Dr. Thomas Freeman Cope, National Research Fellow in Mathematics, Harvard University, Cambridge, Mass.)

Ernst Edward Kummer[1] (1810–1893), who was professor of mathematics in the University of Breslau from 1842 till 1855 and then in the University of Berlin until 1884, made valuable contributions in several branches of mathematics. Among the topics he studied may be mentioned the theory of the hypergeometric (Gaussian) series, the Riccati equation, the question of the convergency of series, the theory of complex numbers, and cubic and biquadratic residues. He was the creator of ideal prime factors of complex numbers and studied intensively surfaces of the fourth order and, in particular, the surfaces which bear his name.

In the following paper which appears in the original in Crelle's *Journal für die reine und angewandte Mathematik* (Vol. 35, pp. 319–326, 1847), Kummer introduces the notion of ideal prime factors of complex numbers, by means of which he was able to restore unique factorization in a field where the fundamental theorem of arithmetic does not hold. Although Kummer's theory has been largely supplanted by the simpler and more general theory of Dedekind, yet the ideas he introduced were of such importance that no less an authority than Professor E. T. Bell is responsible for the statement that[2] "Kummer's introduction of ideals into arithmetic was beyond all dispute one of the greatest mathematical advances of the nineteenth century." For the position of Kummer's theory in the theory of numbers, the reader is referred to the article by Professor Bell from which the above quotation is taken.

On the Theory Of Complex Numbers

(By Professor Kummer of Breslau.)

(Abstract of the *Berichten der Königl. Akad. der Wiss. zu Berlin*, March 1845.)

I have succeeded in completing and in simplifying the theory of those complex numbers which are formed from the higher roots of unity and which, as is well known, play an important rôle in cyclotomy and in the study of power residues and of forms of higher degree; this I have done through the introduction of a peculiar kind of imaginary divisors which I call *ideal complex*

[1] For a short biographical sketch, see D. E. Smith, *History of Mathematics*, Vol. I, pp. 507–508, Boston, 1923.

[2] *American Mathematical Monthly*, Vol. 34, pp. 66.

numbers and concerning which I take the liberty of making a few remarks.

If α is an imaginary root of the equation $\alpha^\lambda = 1$, λ a prime number, and a, a_1, a_2, etc. whole numbers, then $f(\alpha) = a + a_1\alpha + a_2\alpha^2 + \ldots + a_{\lambda-1}\alpha^{\lambda-1}$ is a complex whole number. Such a complex number can either be broken up into factors of the same kind or such a decomposition is not possible. In the first case, the number is a composite number; in the second case, it has hitherto been called a complex prime number. I have observed, however, that, even though $f(\alpha)$ cannot in any way be broken up into complex factors, it still does not possess the true nature of a complex prime number, for, quite commonly, it lacks the first and most important property of prime numbers; namely, that the product of two prime numbers is divisible by no other prime numbers. Rather, such numbers $f(\alpha)$, even if they are not capable of decomposition into complex factors, have nevertheless the nature of composite numbers; the factors in this case are, however, not actual but ideal complex numbers. For the introduction of such ideal complex numbers, there is the same, simple, basal motive as for the introduction of imaginary formulas into algebra and analysis; namely, the decomposition of integral rational functions into their simplest factors, the linear. It was, moreover, such a desideratum which prompted Gauss, in his researches on biquadratic residues (for all such prime factors of the form $4m + 1$ exhibit the nature of composite numbers), to introduce for the first time complex numbers of the form $a + b\sqrt{-1}$.

In order to secure a sound definition of the true (usually ideal) prime factors of complex numbers, it was necessary to use the properties of prime factors of complex numbers which hold in every case and which are entirely independent of the contingency of whether or not actual decomposition takes place: just as in geometry, if it is a question of the common chords of two circles even though the circles do not intersect, one seeks an actual definition of these ideal common chords which shall hold for all positions of the circles. There are several such permanent properties of complex numbers which could be used as definitions of ideal prime factors and which would always lead to essentially the same result; of these, I have chosen *one* as the simplest and the most general.

If p is a prime number of the form $m\lambda + 1$, then it can be represented, in many cases, as the product of the following $\lambda - 1$ complex factors: $p = f(\alpha)\cdot f(\alpha^2)\cdot f(\alpha^3)\ldots f(\alpha^{\lambda-1})$; when, however, a decomposition into actual complex prime factors is not possible, let ideals make their appearance in order to bring this about. If $f(\alpha)$ is an actual complex number and a prime factor of p, it has the property that, if instead of the root of the equation $\alpha^\lambda = 1$ a definite root of the congruence $\xi^\lambda \equiv 1$, mod. p, is substituted, then $f(\xi) \equiv 0$, mod. p. Hence too if the prime factor $f(\alpha)$ is contained in a complex number $\Phi(\alpha)$, it is true that $\Phi(\xi) \equiv 0$, mod. p; and conversely, if $\Phi(\xi) \equiv 0$, mod. p, and p is factorable into $\lambda - 1$ complex prime factors, then $\Phi(\alpha)$ contains the prime factor $f(\alpha)$. Now the property $\Phi(\xi) \equiv 0$, mod. p, is such that it does not depend in any way on the factorability of the number p into prime factors; it can accordingly be used as a definition, since it is agreed that the complex number $\Phi(\alpha)$ shall contain the ideal prime factor of p which belongs to $\alpha = \xi$, if $\Phi(\xi) \equiv 0$, mod. p. Each of the $\lambda - 1$ complex prime factors of p is thus replaced by a congruence relation. This suffices to show that complex prime factors, whether they be actual or merely ideal, give to complex numbers the same definite character. In the process given here, however, we do not use the congruence relations as the definitions of ideal prime factors because they would not be sufficient to represent several equal ideal prime factors of a complex number, and because, being too restrictive, they would yield only ideal prime factors of the real prime numbers of the form $m\lambda - 1$.

Every prime factor of a complex number is also a prime factor of every real prime number q, and the nature of the ideal prime factors is, in particular, dependent on the exponent to which q belongs for the modulus λ. Let this exponent be f, so that $q^f \equiv 1$, mod. λ, and $\lambda - 1 = e \cdot f$. Such a prime number q can never be broken up into more than e complex prime factors which, if this decomposition can actually be carried out, are represented as linear functions of the e periods of each set of f terms. These periods of the roots of the equation $\alpha^\lambda = 1$, I denote by $\eta, \eta_1, \eta_2, \ldots \eta_{e-1}$; and indeed in such an order that each goes over into the following one whenever α is transformed into α^γ, where γ is a primitive root of λ. As is well known, the periods are the e roots of an equation of the eth degree; and this equation, considered as a congruence for the modulus q, has always e real congruential roots which I denote by $u, u_1, u_2, \ldots u_{e-1}$ and take in an order corresponding to that of the periods, for which, besides the congruence of the eth degree, still other easily found congruences may be used. If now the complex number $c'\eta + c_1'\eta_1 + c_2'\eta_2 +$

$\ldots + c'_{e-1}\eta_{e-1}$, constructed out of periods, is denoted shortly by $\Phi(\eta)$, then among the prime numbers q which belong to the exponent f, there are always such that can be brought into the form

$$q = \Phi(\eta)\Phi(\eta_1)\Phi(\eta_2)\ldots\Phi(\eta_{e-1}),$$

in which, moreover, the e factors never admit a further decomposition. If one replaces the periods by the congruential roots corresponding to them, where a period can arbitrarily be designated to correspond to a definite congruential root, then one of the e prime factors always becomes congruent to zero for the modulus q. Now if any complex number $f(\alpha)$ contains the prime factor $\Phi(\eta)$, it will always have the property, for $\eta = u_k$, $\eta_1 = u_{k+1}$, $\eta_2 = u_{k+2}$, etc., of becoming congruent to zero for the modulus q. This property (which implies precisely f distinct congruence relations, the development of which would lead too far) is a permanent one even for those prime numbers q which do not admit an actual decomposition into e complex prime factors. It could therefore be used as a definition of complex prime factors; it would, however, have the defect of not being able to express the equal ideal prime factors of a complex number.

The definition of ideal complex prime factors which I have chosen and which is essentially the same as the one described but is simpler and more general, rests on the fact that, as I prove separately, one can always find a complex number $\psi(\eta)$, constructed out of periods, which is of such a nature that $\psi(\eta)\psi(\eta_1)\psi(\eta_2)\ldots\psi(\eta_{e-1})$ (this product being a whole number) is divisible by q but not by q^2. This complex number $\psi(\eta)$ has always the abovementioned property, namely, that it is congruent to zero, modulo q, if for the periods are substituted the corresponding congruential roots, and therefore $\psi(\eta) \equiv 0$, mod. q, for $\eta = u$, $\eta_1 = u_1$, $\eta_2 = u_2$, etc. I now set $\psi(\eta_1)\psi(\eta_2)\ldots\psi(\eta_{e-1}) = \Psi(\eta)$ and define ideal prime numbers in the following manner:—

If $f(\alpha)$ has the property that the product $f(\alpha).\Psi(\eta_r)$ is divisible by q, this shall be expressed as follows: $f(\alpha)$ contains the ideal prime factor of q which belongs to $u = \eta_r$. Furthermore, if $f(\alpha)$ has the property that $f(\alpha).(\Psi(\eta_r))^\mu$ is divisible by q^μ but $f(\alpha)(\Psi(\eta_r))^{\mu+1}$ is not divisible by $q^{\mu+1}$, this shall be described thus: $f(\alpha)$ contains the ideal prime factor of q which belongs to $u = \eta_r$, exactly μ times.

It would lead too far if I should develop here the connection and the agreement of this definition with those given by congruence relations as described above; I simply remark that the relation: $f(\alpha)\Psi(\eta_r)$ divisible by q, is completely equivalent to f distinct congruence relations, and that the relation: $f(\alpha)(\Psi(\eta_r))^\mu$ divisible by q^μ, can always be entirely replaced by $u \cdot f$ congruence relations. The whole theory of ideal complex numbers which I have already perfected and of which I here announce the principal theorems, is a justification of the definition given as well as of the nomenclature adopted. The principal theorems are the following:

The product of two or more complex numbers has exactly the same ideal prime factors as the factors taken together.

If a complex number (which is a product of factors) contains all the e prime factors of q, it is also divisible by q itself; if, however, it does not contain some one of these e ideal prime factors, it is not divisible by q.

If a complex number (in the form of a product) contains all the e ideal prime factors of q and, indeed, each at least μ times, it is divisible by q^μ.

If $f(\alpha)$ contains exactly m ideal prime factors of q, which may all be different, or partly or wholly alike, then the norm $Nf(\alpha) = f(\alpha)f(\alpha^2)\ldots f(\alpha^{\lambda-1})$ contains exactly the factor q^{mf}.

Every complex number contains only a finite, determinate number of ideal prime factors.

Two complex numbers which have exactly the same ideal prime factors differ only by a complex unit which may enter as a factor.

A complex number is divisible by another if all the ideal prime factors of the divisor are contained in the dividend; and the quotient contains precisely the excess of the ideal prime factors of the dividend over those of the divisor.

From these theorems it follows that computation with complex numbers becomes, by the introduction of ideal prime factors, entirely the same as computation with integers and their real integral prime factors. Consequently, the grounds for the complaint which I voiced in the *Breslauer Programm zur Jubelfeier der Universität Königsberg S.* 18, are removed:—

It seems a great pity that this quality of real numbers, namely, that they can be resolved into prime factors which for the same number are always the same, is not shared by complex numbers; if now this desirable property were part of a complete doctrine, the effecting of which is as yet beset with great difficulties, the matter could easily be resolved and brought to a successful conclusion. Etc. One sees therefore that ideal prime factors disclose the inner nature of complex numbers, make them transparent, as it were, and show their inner crystalline structure. If, in particular, a complex number is given merely in the form $a + a_1\alpha + a_2\alpha^2 + \ldots + a_{\lambda-1}\alpha^{\lambda-1}$, little can be asserted about it until one has determined, by means of its ideal prime factors (which in such a case can always be found by direct methods), its simplest qualitative properties to serve as the basis of all further arithmetical investigations.

Ideal factors of complex numbers arise, as has been shown, as factors of actual complex numbers: hence ideal prime factors multiplied with others suitably chosen must always give actual complex numbers for products. This question of the combination of ideal factors to obtain actual complex numbers is, as I shall show as a consequence of the results which I have already found, of the greatest interest, because it stands in an intimate relationship to the most important sections of number theory. The two most important results relative to this question are the following:

There always exists a finite, determinate number of ideal complex multipliers which are necessary and sufficient to reduce all possible ideal complex numbers to actual complex numbers.[1]

Every ideal complex number has the property that a definite integral power of it will give an actual complex number.

I consider now some more detailed developments from these two theorems. Two ideal complex numbers which, when multiplied by one and the same ideal number, form actual complex numbers, I shall call *equivalent* or of the same class, because this investigation of actual and ideal complex numbers is identical with the classification of a certain set of forms of the $\lambda - 1$st degree and in $\lambda - 1$ variables; the principal results relative to this classification have been found by Dirichlet but not yet published so that I do not know precisely whether or not his principle of classification coincides with that resulting from the theory of complex numbers. For example, the theory of a form of the second degree in two variables with determinant, however, a prime number λ, is closely interwoven with these investigations, and our classification in this case coincides with that of Gauss but not with that of Legendre. The same considerations also throw great light upon Gauss's classification of forms of the second degree and upon the true basis for the differentiation between *Aequivalentia propria et impropria*,[2]

[1] A proof of this important theorem, although in far less generality and in an entirely different form, is found in the dissertation: L. Kronecker, *De unitatibus complexis*, Berlin, 1845.

[2] [i. *e.*, proper and improper equivalence.]

which, undeniably, has always an appearance of impropriety when it presents itself in the *Disquisitiones arithmeticae*. If, for example, two forms such as $ax^2 + 2bxy + cy^2$ and $ax^2 - 2bxy + cy^2$, or $ax^2 + 2bxy + cy^2$ and $cx^2 + 2bxy + ay^2$, are considered as belonging to different classes, as is done in the above-mentioned work, while in fact no essential difference between them is to be found; and if on the other hand Gauss's classification must notwithstanding be admitted to be one arising for the most part out of the very nature of the question: then one is forced to consider forms such as $ax^2 + 2bxy + cy^2$ and $ax^2 - 2bxy + cy^2$ which differ from each other in outward appearance only, as merely representative of two new but essentially different concepts of number theory. These however, are in reality nothing more than two different ideal prime factors which belong to one and the same number. The entire theory of forms of the second degree in two variables can be thought of as the theory of complex numbers of the form $x + y\sqrt{D}$ and then leads necessarily to ideal complex numbers of the same sort. The latter, however, classify themselves according to the ideal multipliers which are necessary and sufficient to reduce them to actual complex numbers of the form $x + y\sqrt{D}$. Because of this agreement with the classification of Gauss, ideal complex numbers thus constitute the true basis for it.

The general investigation of ideal complex numbers presents the greatest analogy with the very difficult section by Gauss: *De compositione formarum*, and the principal results which Gauss proved for quadratic forms, pp. 337 and following, hold true also for the combination of general ideal complex numbers. Thus there belongs to every class of ideal numbers another class which, when multiplied by the first class, gives rise to actual complex numbers (here the actual complex numbers are the analogue of the *Classis principalis*).[1] Likewise, there are classes which, when multiplied by themselves, give for the result actual complex numbers (the *Classis principalis*), and these classes are therefore *ancipites;*[2] in particular, the *Classis principalis* itself is always a *Classis anceps*. If one takes an ideal complex number and raises it to powers, then in accordance with the second of the foregoing theorems, one will arrive at a power which is an actual complex number; if h is the smallest number for which $(f(\alpha))^h$ is an actual

[1] [Principal class.]
[2] [Dual, or of a double nature.]

complex number, then $f(\alpha)$, $(f(\alpha))^2$, $(f(\alpha))^3, \ldots (f(\alpha))^h$ all belong to different classes. It now may happen that, by a suitable choice of $f(\alpha)$, these exhaust all existing classes: if such is not the case, it is easy to prove that the number of classes is at least always a multiple of h. I have not gone deeper yet into this domain of complex numbers; in particular, I have not undertaken an investigation of the exact number of classes because I have heard that Dirichlet, using principles similar to those employed in his famous treatise on quadratic forms, has already found this number. I shall make only one additional remark about the character of ideal complex numbers, namely, that by the second of the foregoing theorems they can always be considered and represented as definite roots of actual complex numbers, that is, they always take the form $\sqrt[h]{\Phi(\alpha)}$ where $\Phi(\alpha)$ is an actual complex number and h an integer.

Of the different applications which I have already made of this theory of complex number, I shall refer only to the application to cyclotomy to complete the results which I have already announced in the above-mentioned *Programm*. If one sets

$$(\alpha, x) = x + \alpha x^g + \alpha^2 x^{g^2} + \ldots + \alpha^{p-2} x^{g^{p-2}},$$

where $\alpha^\lambda = 1$, $x^p = 1$, $p = m\lambda + 1$, and g is a primitive root of the prime number p, then it is well known that $(\alpha, x)^\lambda$ is a complex number independent of x and formed from the roots of the equation $\alpha^\lambda = 1$. In the *Programm* cited, I have found the following expression for this number, under the assumption that p can be resolved into $\lambda - 1$ actual complex prime factors, one of which is $f(\alpha)$:

$$(\alpha, x)^\lambda = \pm \alpha^h f^{m_1}(\alpha) \cdot f^{m_2}(\alpha^2) \cdot f^{m_3}(\alpha^3) \ldots f^{m_{\lambda-1}}(\alpha^{\lambda-1}),$$

where the power-exponents m_1, m_2, m_3, etc. are so determined that the general m_K, positive, is less than λ and $k \cdot m_k \equiv 1$, mod. λ. Exactly the same simple expression holds in complete generality, as can easily be proved, even when $f(\alpha)$ is not the actual but only the ideal prime factor of p. In order, however, in the latter case, to maintain the expression for $(\alpha, x)^\lambda$ in the form for an actual complex number, one need only represent the ideal $f(\alpha)$ as a root of an actual complex number, or apply one of the methods (although indirect) which serve to represent an actual complex number whose ideal prime factors are given.

CHEBYSHEV (TCHEBYCHEFF)

On the Totality of Primes

(*Translated from the French by Professor J. D. Tamarkin, Brown University, Providence, Rhode Island.*)

Pafnuty Lvovich Chebyshev (Tchebycheff, Tchebytcheff) was born on May 14, 1821, and died on Nov. 26, 1894. He is one of the most prominent representatives of the Russian mathematical school. He made numerous important contributions to the theory of numbers, algebra, the theory of probabilities, analysis, and applied mathematics. Among the most important of his papers are the two memoirs of which portions are here translated:

1. "Sur la totalité des nombres premiers inférieurs à une limite donnée," *Mémoires presentés à l'Académie Impériale des Sciences de St.-Pétersbourg par divers savants et lus dans ses assemblées*, Vol. 6, pp. 141–157, 1851 (Lu le 24 Mai, 1848); *Journal de Mathématiques pures et appliquées*, (1) Vol. 17, pp. 341–365, 1852; *Oeuvres*, Vol. 1, pp. 29–48, 1899.

2. "Mémoire sur les nombres premiers," *ibid.*, Vol. 7, pp. 15–33, 1854 (lu le 9 Septembre, 1850), *ibid.*, pp. 366–390, *ibid.*, pp. 51–70.

These memoirs represent the first definite progress after Euclid in the investigation of the function $\phi(x)$ which determines the totality of prime numbers less than the given limit x. The problem of finding an asymptotic expression for $\phi(x)$ for large values of x attracted the attention and efforts of some of the most brilliant mathematicians such as Legendre, Gauss, Lejeune-Dirichlet, and Riemann.

Gauss (1791, at the age of fourteen) was the first to suggest, in a purely empirical way, the asymptotic formula $\frac{x}{\log x}$ for $\phi(x)$. (*Werke*, Vol. X_1, p. 11, 1917.) Later on (1792–1793, 1849), he suggested another formula $\int_2^x \frac{dx}{\log x}$, of which $\frac{x}{\log x}$ is the leading term (Gauss's letter to Encke, 1849, *Werke*, Vol. II, pp. 444–447, 1876). Legendre, being, of course, unaware of Gauss's results, suggested another empirical formula $\frac{x}{A \log x + B}$ (*Essai sur la théorie des nombres*, 1st ed., pp. 18–19, 1798) and specified the constants A and B as $A = 1$, $B = -1.08366$ in the second edition of the *Essai* (pp. 394–395, 1808). Legendre's formula, which Abel quoted as "the most marvelous in mathematics" (letter to Holmboe, *Abel Memorial*, 1902, Correspondence, p. 5), is correct up to the leading term only. This fact was recognized by Dirichlet ("Sur l'usage des séries infinies dans la théorie des nombres," *Crelle's Journal*, Vol. 18, p. 272, 1838, in his remark written on the copy presented to Gauss. *Cf.* Dirichlet, *Werke*, Vol. 1, p. 372, 1889). In this note