Predicativity, the Russell-Myhill Paradox, and Church's Intensional Logic

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Introduction

- Just as there is the Russell paradox about sets, so there is the Russell-Myhill paradox of propositions.

- While predicativity has been well-explored as a response to the Russell paradox of sets, there seems to have been no previous attempt to set out a predicative solution to the Russell-Myhill paradox of propositions.

- The aim of this talk is to do that, within the framework of Church's intensional logic.

Introduction, continued

- In previous work, we've focused on building models of fragments of Frege's set theory:

Sean Walsh. Comparing Hume's Principle, Basic Law V and Peano Arithmetic. *Annals of Pure and Applied Logic*, 163:1679–1709, 2012.

Sean Walsh. Fragments of Frege's Grundgesetze and Gödel's constructible universe. *The Journal of Symbolic Logic*, forthcoming. arXiv:1407.3861.

- Today we explain how such set theory can be seen to arise through Church's intensional logic. This talk follows closely:

Sean Walsh. Predicativity, the Russell-Myhill paradox, and Church's Intensional Logic. *The Journal of Philosophical Logic*, pages 1–50, 2015. http://arxiv.org/abs/1506.02206.

Outline

- *I. Introduction*
- *II. Neutral Core of Church's Intensional Logic*
- *III. Formalized Version of Russell-Myhill Paradox*
- *IV. Predicative Response to Russell-Myhill*
- *V. The Problem of Many Non-Extensions*
- *VI. Conclusions and Further Questions*

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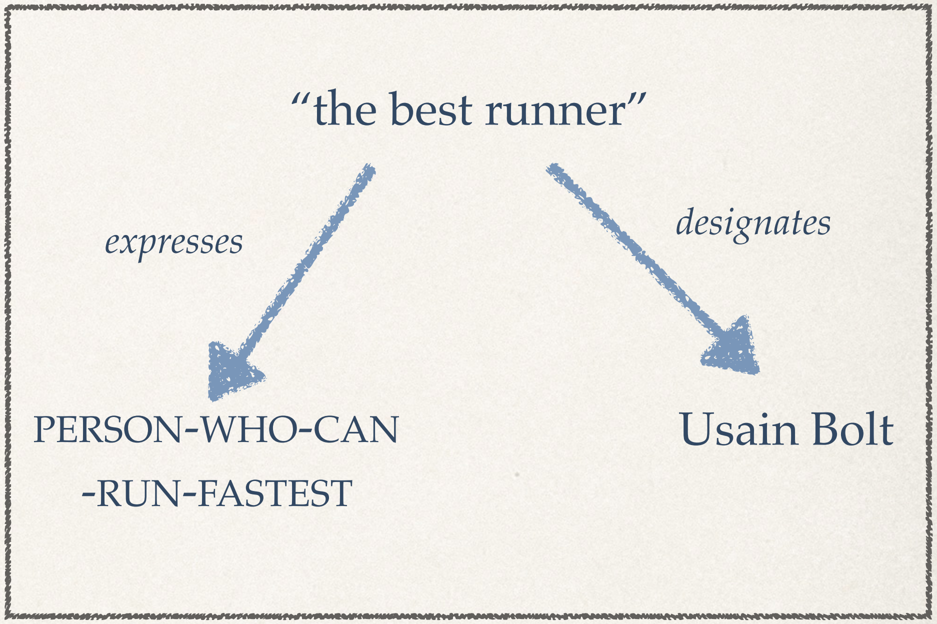
Frege's Sense-Reference Distinction

- The intensional logic of Church is an attempt to axiomatize Frege's sense-reference distinction.
- Frege thought that words not only designate their referent, but also express their sense.
- On this view, our words bear two semantic relations to non-linguistic entities, namely they bear the *designation relation* to their referents and they bear the *expression relation* to their senses.



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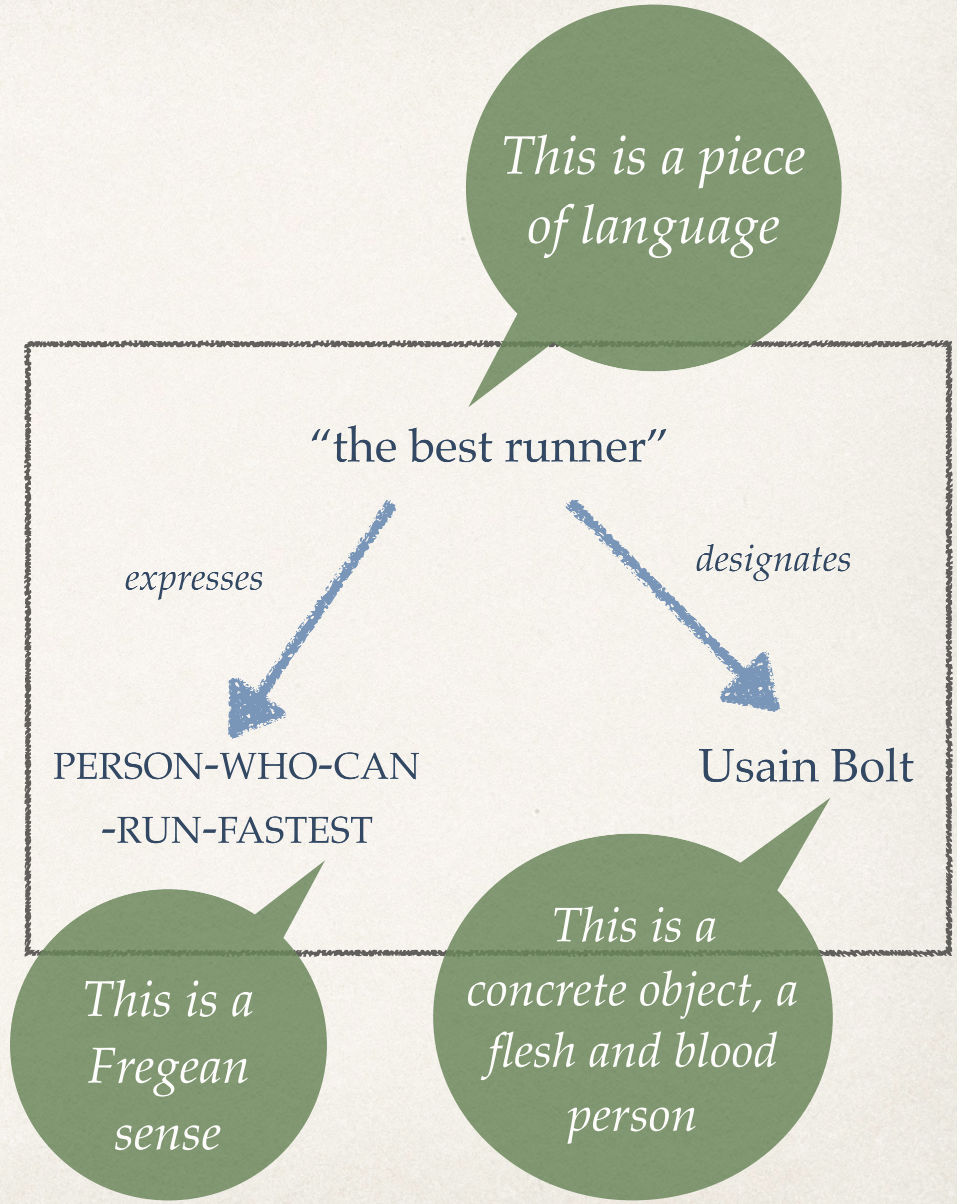
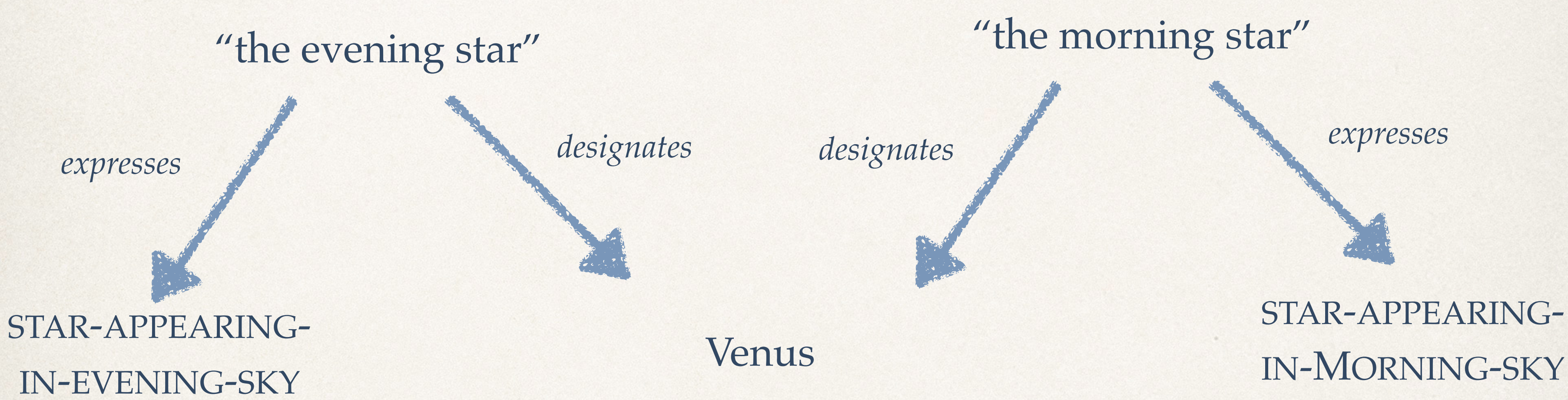


Visualizing the Morning-Star Evening Star Example



Visualizing the Morning-Star Evening Star Example

This is a piece of language

"the evening star"

expresses

designates

STAR-APPEARING-IN-EVENING-SKY

This is a Fregean sense

Venus

This is a concrete object, an actual planet

This is a piece of language

"the morning star"

designates

expresses

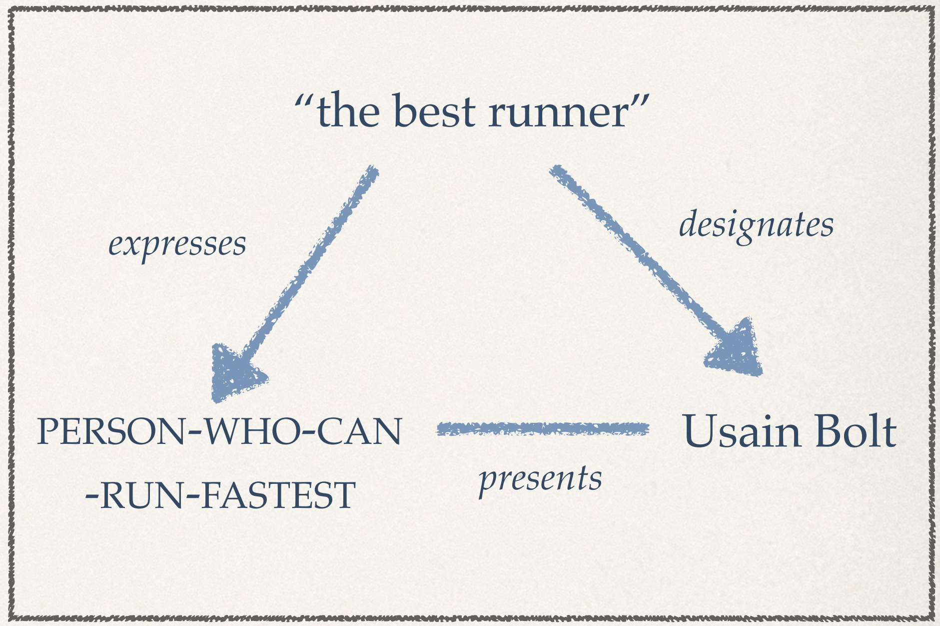
STAR-APPEARING-IN-MORNING-SKY

This is a Fregean sense

Church's Presentation Relationship

- Church observed that Frege is committed to a relationship between the abstract Fregean sense expressed by a linguistic expression and the entity which is denoted by that linguistic expression.

- This relation is called the *presentation* relation in the literature, and one says sense *s presents* denotation *d* and one writes $\Delta(s,d)$ precisely in the circumstance where there is a linguistic expression which expresses *s* and denotes *d*.



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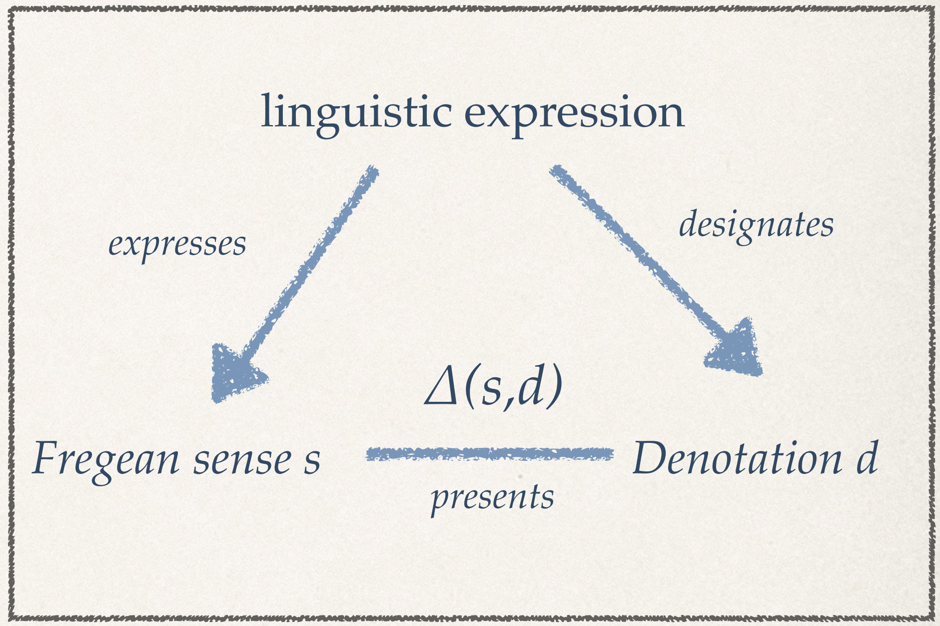

Church’s Method and Goals

- Church proceeded by *axiomatizing* the presentation relationship $\Delta(s,d)$.

- Why axiomatize? He thought that this would be a way to dissipate skepticism about Fregean sense.

- To the right is a quotation from Church from 1943.

- “There remains the important task, which has never been approached, of constructing a formalized semantical system which shall take account of both kinds of meaning, the relation between a name and its denotation, and the relation between a name and its sense. […] [¶] […] Ultimately it is only on the basis of their inclusion in an adequate system of this kind that such otherwise indefensibly vague ideas as ‘understanding’ of an expression, ‘attribute,' ‘objectiver Inhalt des Denkens,' may be regarded as logically significant.”

The Types of Church's System

- Church's system is a typed system, with types defined as:
  (i) there is a type *e* of objects.
  (ii) there is a type *t* of truth-values.
  (iii) if *a*, *b* are types, then there is a type *ab* of functions from entities of type *a* to entities of type *b*.
  (iv) if *a* is a type, then there is a type *a′* for senses of entities of type *a*.

- *Example 1*: type *et* is the type reserved for functions from objects to truth-values, i.e. Fregean concepts.

- *Example 2*: type *t′* is the type reserved for senses of truth-values, i.e. propositions.

- Example 3: type *t′t* is the type of functions from propositions to truth-values, i.e. collections of propositions.

The Types of Church's System, Continued

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- Since the system is typed, there is thus not just one presentation relation Δ, but there is a presentation relation Δ_a for each type *a*.

- For instance, Δ_e is the relation which a Fregean sense of an object bears to the object, while Δ_t is the relation which a proposition bears to its truth-value.

Church's Axioms

- This first axiom simply says that the presentation relationship is functional:

Typed Sense Determines Reference: $(\Delta_a(s, d_0) \,\&\, \Delta_a(s, d_1)) \Rightarrow d_0 = d_1$

This warrants us in writing $\Delta_a(s)=d$ instead of $\Delta_a(s,d)$.

Church's Axioms, Continued

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$$\textit{Typed Sense Determines Reference: } (\Delta_a(s, d_0)\ \&\ \Delta_a(s, d_1)) \Rightarrow d_0 = d_1$$

- The second axiom says that composition on the side of sense matches up to composition on the side of reference:

$$\textit{Typed Composition: } [\Delta_{ab}(f') = f\ \&\ \Delta_a(x') = x] \Longrightarrow \Delta_b(f'\langle x'\rangle) = f(x)$$

For instance, *Planet(Venus)=true*, and hence one should have that *HEAVENLY-BODY⟨STAR-APPEARING-IN-MORNING-SKY⟩* presents the true.

- In the composition axiom, the notation $f'\langle x'\rangle$ is a primitive intensional application function on senses.

Church's Axiom of Type Reduction

- Church himself postulated a further axiom:
  Church's Axiom of Type-Reduction:
  $(ab)' = a'b'$

- This says that the senses of functions are functions from senses to senses.

- Advantage: can interpret intensional application $f'\langle x'\rangle$ as extensional application $f'(x')$.

- Disadvantages:

  - Dummett thought that this was inconsistent with the idea that we may learn the senses of complete sentences before we learn the senses of their constituent components.

  - Bealer writes: "Joy, the shape of my hand, the aroma of coffee— these are not functions."

The Neutral Core of Church's Intensional Logic

- Hence, if we reject Church's Axiom of Type Reduction, we are left with:
- The *Neutral Core of Church's Intensional Logic*:
- 1. Typed Sense Determines Reference Axiom
- 2. Typed Composition Axiom

The Neutral Core of Church's Intensional Logic, Continued

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  Kaplan noted in 1975 that usual possible worlds semantics naturally yield models of the Neutral Core.
- For, we may interpret type *a'* as the set of functions from worlds to entities of type *a*. And then we can define:

$$\Delta_a(f') = f'(w_0), \qquad (f'\langle x'\rangle)(w) = (f'(w))(x'(w))$$

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The Usual Version of The Russell-Myhill Paradox

- Let C be a collection of propositions. Let $\iota(C)$ be the proposition "Everything is in C".
- It seems this is an injection from collections of propositions to propositions. Suppose $\iota(C) = \iota(D)$. Then these two propositions are the same, and so the senses of C and D are the same, and hence C and D are identical as collections.
- But then this contradicts type-theoretic version of Cantor's theorem.

The Usual Version of The Russell-Myhill Paradox, Continued

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- But then this contradicts type-theoretic version of Cantor's theorem.
- Three reasons to be dissatisfied with this version of the paradox.
- First, not clear what formal system this is supposed to be rendered in.
- Second, in the italicized part of the argumentation, not entirely clear what the senses of C and D are— after all, we just started out with two collections of propositions.
- Third, in underlined part, we use a 'converse compositional' principle: if the wholes are the same, so are the parts.

Alternative Formalization of Russell-Myhill Paradox

- The following are jointly inconsistent against the background of the neutral core of Church's intensional logic:
  - 1. The Surjectivity Axiom
  - 2. The Senses are Objects Axiom
  - 3. Propositions As Fine-Grained as Objects Axiom
  - 4. The Typed Choice Schema
- Let's say a word briefly about each of these, before going on to show the inconsistency.

The Surjectivity Axiom

Surjectivity Axiom: for each type a and each element f of type a, there is element f' of type a' such that $\Delta_a(f') = f$.

- The immediate warrant for this axiom is that there is simply no other way to formalize the Russell-Myhill paradox.
- For, consider again how it opens: "for each collection of propositions C, consider the proposition $\iota(C)$ expressed by the sentence 'every proposition in C is true.' "
- We accordingly need some way to move from *any* collection of propositions to a proposition.
- It seems that any way in which we do this will take a collection of propositions, take a sense or intension which presents this collection, and build a proposition based off of this sense.

The Senses are Objects Axiom

Senses are Objects Axiom: for each type a and each element f' of type a', there is element x of type e such that $f' = x$.

- The primary reason for this follows from elementary reflections on traditional interpretations of the notion of Fregean sense.
  - Dummett (and Tichy) suggested that we might understand senses as certain kinds of procedures or algorithms, a "route to reference." These can be treated as indexes for Turing machines, a natural number.
  - Another traditional interpretation of Fregean sense is that of definite descriptions. But these can be treated as bits of syntax, something like a Gödel number for a formula.

Propositions As Fine-Grained as Objects Axiom

Propositions as Fine-Grained as Objects Axiom: there is an injection χ from entities of type e to entities of type t'.

- This axiom says that there's an injection from objects to propositions.
- One plausible case for this axiom might be made from the assumption that (i) our language is ample enough to distinguish different objects from one another and (ii) propositions are organized roughly after the manner of the sentences which express them.

The Typed Choice Schema

(Typed Choice Schema). The *typed choice schema* consists of all the axioms

$$\forall\, z_1, \ldots, z_k\ [[\forall\, x\ \exists\, y\ \varphi(x, y, z_1, \ldots, z_k)] \rightarrow \exists\, h\ [\forall\, x\ \varphi(x, h(x), z_1, \ldots, z_k)]]$$

where $\varphi(x, y, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, y of type b, while h is a variable of type ab which does not appear free in φ.

- This holds on "standard models" by virtue of the axiom of choice in the ambient metatheory.
- It implies the usual kinds of comprehension schema, which roughly says that any formula determines a higher-order entity of the appropriate type.

Recall: Alternative Formalization of Russell-Myhill Paradox

- The following are jointly inconsistent against the background of the neutral core of Church's intensional logic:
  - 1. The Surjectivity Axiom
  - 2. The Senses are Objects Axiom
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Proof of Inconsistency

- 1. By the Surjectivity Axiom, for any collection C of propositions, there is a sense C' which presents it.
- 2. By the Senses are Objects axiom, for any collection C of propositions, there is an object x and there is sense C' which presents C and which is equal to x.
- 3. By the Typed Choice Schema, there is a map $C \mapsto \delta(C)$ from collections C of propositions to objects $\delta(C)$ such that there is a sense C' which presents C and which is equal to $\delta(C)$.
- 4. Then δ is an injection from collections of propositions to objects. For, suppose that $\delta(C)=\delta(D)$. Then there is sense C', D' such that C' presents C, D' presents D, and $C'=\delta(C)=\delta(D)=D'$. By the Typed Sense Determines Reference Axiom, we have that $C=\Delta_{t't}(C') = \Delta_{t't}(D')=D$.
- 5. By composing δ with the injection χ from objects to propositions postulated by the Propositions As Fine-Grained as Objects Axiom, we obtain an injection from collections of propositions to propositions, which contradicts the type-theoretic version of Cantor's theorem, which is derivable from the comprehension schema (which is in turn derivable from the Typed Choice Schema).

Which axioms fail in the possible worlds models of the Neutral Core?

- These models assign the domains D_a to each type a:

$$D_e = E, \quad D_t = \{0,1\}, \quad D_{ab} = D_b^{D_a} = \{f : D_a \to D_b\}, \quad D_{a'} = D_a^W = \{f : W \to D_a\}$$

- Suppose $|P(W)| < |E|$. Then one has

$$|D_{t'}| = |P(W)| < |E| = |D_e|.$$

Then Propositions As Fine-Grained as Objects axiom fails.

- Suppose $|P(W)| \geq |E|$. Then one has that:

$$|D_{(t't)'}| \geq |D_{t't}| > |D_{t'}| = |P(W)| \geq |E| = |D_e|$$

Then Senses are Objects axiom fails.

Anderson and the Failure of Surjectivity

- In the work of C. Anthony Anderson, one finds a comparatively accommodating way to reject the Surjectivity Axiom.
- Just like in the "typed" approaches to the liar paradox, Anderson suggests that there is a hierarchy of presentation relations $\Delta_a^{(1)}, \Delta_a^{(2)}, \Delta_a^{(3)}, \ldots$
- Thus while every entity might be n-th order presented for some $n \geq 1$, the thought is that there is no single $n \geq 1$ such that every entity is n-th order presented.

C. Anthony Anderson. Some new axioms for the logic of sense and denotation: alternative (0). *Noûs*, 14(2):217–234, 1980.

C. Anthony Anderson. General intensional logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Vol. II: Extensions of Classical Logic*, volume 165 of *Synthese Library*, pages 355–385. Reidel, Dordrecht, 1984.

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Predicativity

- In current mathematical logic, "predicativity" is associated with restrictions on the comprehension schema.
- The comprehension schema says that formulas determine concepts. More formally:

(Concept Comprehension Schema). The *concept comprehension schema* consists of all the axioms $\forall\, z_1, \ldots, z_k\; \exists\, h\; \forall\, x\; (h(x) = 1 \leftrightarrow \psi(x, z_1, \ldots, z_k))$, where $\psi(x, z_1, \ldots, z_k)$ is a formula with all free variables displayed and with free variable x of type a, while h is a variable of type at which does not appear free in ψ.

- It is implied by the Typed Choice Schema. Hence, if you have reasons to reject comprehension, you have a potential way out of the paradox.

Poincaré and Weyl on the Comprehension Schema

- They drew attention to the fact that higher-order definitions are not in general preserved when one keeps the first-order domain fixed but expands the range of the higher-order quantifiers.

- Some care has to be exercised in making such a historical attribution, since Poincaré was writing before the comprehension schema was formalized.

- But Weyl is rightly credited as the first to study mathematics in the presence of restrictions on the comprehension schema, and it seems to me that he's making much the same philosophical point as Poincaré in certain places.

- For reference, the relevant quotations from Poincaré and Weyl are on the next slides.

Quotations from Poincaré

De là une distinction entre deux espèces de classifications, appliquables aux éléments des collections infinies; les classifications prédicatives, qui ne peuvent être bouleverées par l'introduction de nouveaux élements; les classifications *non prédicatives* que l'introduction des éléments nouveaux oblige à remanier sans cesse.

p. 463 of: Henri Poincaré. La logique de l'infini. *Revue de métaphysique et de morale*, 17:461–482, 1909.

p. 237 of: Gerhard Heinzmann. *Poincaré, Russell, Zermelo et Peano. Textes de la discussion (1906-1912) sur les fondements des mathématiques: des antinomie à la prédicativié.* Blanchard, Paris, 1986.

Ich verstehe darunter folgendes: Jedes Zuordnungsgesetz setzt eine bestimmte Klassifikation voraus. Ich nenne nun eine Zuordnung prädikativ, wenn die zugehörige Klassifikation prädikativ ist. Eine Klassifikation aber nenne ich prädikativ, wenn sie durch Einfürhung neuer Elemente nicht verändert wird.

p. 47 of: Henri Poincaré. Über transfinite Zahlen. In *Sechs Vorträge über ausgewählte Gegenstände aus der reinen Mathematik und mathematischen Physik*, Mathematische Vorlesungen an der Universität Göttingen, pages 44–48. Teubner, Leipzig/Berlin, 1910.

p. 233 of: Gerhard Heinzmann. *Poincaré, Russell, Zermelo et Peano. Textes de la discussion (1906-1912) sur les fondements des mathématiques: des antinomie à la prédicativié.* Blanchard, Paris, 1986.

Quotation from Weyl

Die Stetigkeit einer Funktion, sahen wir, ist eine *transfinite* Eigenschaft; d.h. die Frage, ob eine mit Hilfe unserer Prinzipien definierte Funktion stetig sei oder nicht, erfordert zu ihrer Entscheidung nicht nur die volle Überblickung der natürlichen Zahlen, sondern ebenso die volle Überblickung derjenigen *Mengen* (genauer: derjenigen vierdimensionalen Mengen natürlicher Zahlen), welche durch Kombinierte Andwendung jener Prinzipien in beliebiger Komplikation entspringen. Nehmen wir die Definitionsprinzipien als ein ≪ *offenes* ≫ System, d.h. behalten uns vor, sie. ev. durch Hinzufügung neuer zu erweitern, so muß im allgemeinen auch die Frage, ob eine gegebene Funktion stetig sei, *offen* bleiben (im Gegensatz zu der Entscheidung in allen *finiten* Fragen): eine Funktion, die gemäß unsern Erklärungen stetig ist, könnte dieser Eigenschaft verlustig gehen, wenn unsere Definitionsprinzipien eine Erweiterung erführen und demgemäß zu den ≪ jetzt ≫ vorhandenen reellen Zahlen weitere hinzuträten, bei deren Bildung die neu hinzugefügten Definitionsprinzipien eine Rolle spielen*) .

pp. 65-66 of: Hermann Weyl. *Das Kontinuum. Kritische Untersuchungen über die Grundlagen der Analysis.* Veit, Leipzig, 1918.

The Stability Argument for Predicativity

- Consider an instance of the comprehension schema:
  $\exists\, h\, \forall\, x\, (\varphi(x) \leftrightarrow h(x)=1)$
  One can think of this h as *the* Φ where we define:
  $\Phi(h) \equiv \forall\, x\, (\varphi(x) \leftrightarrow h(x)=1)$

- If the formula $\varphi(x)$ contains higher-order quantifiers, then whether a given h satisfies the description $\Phi(h)$ may vary with expansions of the range of the higher-order quantifiers.

- However, when the formula $\varphi(x)$ does not itself contain higher-order quantifiers, then whether h satisfies the description $\Phi(h)$ will be stable under expansions of the range of the higher-order quantifiers.

- Thus, if one wants definite descriptions to stably effect reference, then one ought to employ only predicative instances of the comprehension schema.

But in what sense does the range of the higher-order quantifiers expand?

- In my view, the best answer to this is tied to the kinds of positive reasons we can give for the Surjectivity Axiom.

- The best positive reason to believe this axiom flows from a conception of what we're trying to model.

- We’re not trying to model higher-order entities as they are in some abstract inaccessible third realm, but we're trying to model higher-order entities insofar as they fall within our “referential ken”.

- And it's natural to think that our resources for referring to higher-order entities expands over time just as our resources for referring to concrete objects expands over time.

Consistency of the Predicative Response

- Hence, the predicative response to the Russell-Myhill paradox proceeds by denying the full comprehension schema (and hence the full choice schema), and replacing it with a suitable predicative versions.

- Much of the paper is devoted to showing that predicative comprehension and choice are consistent with the axioms:

- 1. The Surjectivity Axiom

- 2. The Senses are Objects Axiom

- 3. Propositions As Fine-Grained as Objects Axiom

- The consistency proof goes through the constructible hierarchy of sets. For details, see the paper.

A Note on Church's Other Axiom

- But the models used to show consistency also model strong forms of another of Church's axioms. Here is the strongest form of it:

Characterization of Intensional Injectivity: A function f of type ab is injective if and only if for any f' of type $(ab)'$ such that $\Delta_{ab}(f') = f$, one has that $f'\langle x'\rangle = f'\langle y'\rangle$ implies $\Delta_a(x') = \Delta_a(y')$.

- Take a non-injective function *f(x)* like "is the father of *x*". Then there is some sense THE FATHER OF which presents the father-of function *f* and which is such that the intension THE FATHER OF ⟨THE G⟩ is the same qua intension as THE FATHER OF ⟨THE H⟩, despite the fact that the person who is THE G is not the same as the person who is THE H.

- As Parsons, Klement, and Anderson have stressed, this is not obviously in keeping with a "fine-grained" theory of intensions.

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The System of Frege's *Grundgesetze*

- The models also allow us to define models of fragments of Frege's *Grundgesetze*.
- This system was a system of higher-order logic plus a type-lowering operator ∂, *the extension operator*, from concepts to objects.
- *Basic Law V* of this system says that the extension operator is injective: $\partial(X)=\partial(Y)$ iff $X=Y$.
- Using the extension operator, one can define a membership relation on objects:

  $$a \in b \text{ iff } \exists\, B\ \partial(B)=b \text{ and } Ba$$

- The *Grundgesetze* system is thus like modern set theory in two respects: it has a membership relation, and it was designed to be able to recover a great deal of ordinary mathematics in one uniform setting.

Problems with Modeling Fragments of Frege's *Grundgesetze*

- With full comprehension, the *Grundgesetze* system is inconsistent.
- But in the last decades, work of Parsons, Heck, and Ferreira-Wehmeier showed that the *Grundgesetze* system is consistent with predicative levels of comprehension.
- But nothing in this earlier work suggested anything like an intended model.
- Moreover, Wehmeier noted that predicative models of the *Grundgesetze* system have the following feature:
  - There are always infinitely many objects that are non-extensions.
- But presumably in certain domains of inquiry (like biology and chemistry), there are only finitely many non-extensions.

Modeling Fragments of Frege's *Grundgesetze*

- These problems can be partially addressed by looking at Frege's *Grundgesetze* through the lens of Church's intensional logic.
- Let's define a *sense-selecting* extension operator to be an extension operator such that *the extension of a concept is a sense of that concept*.
- Such exist within our models of Church's intensional logic.
- So why are there many non-extensions in these models of the *Grundgesetze*?
- Because the extension of a concept is but *a single* sense of that concept.
- And part of the explanatory power of Fregean sense resides in the idea that any given referent may be presented by any number of senses.

But is that really what sets are?

- Again, define a *sense-selecting* extension operator to be an extension operator such that *the extension of a concept is a sense of that concept*.

- Recall how we define membership in terms of the extension operator:
  $a \in b$ *iff* $\exists B\ \partial(B)=b$ *and* Ba

- Does this track any prior experience with sets? Yes and no.

- Consider the set
  $b = \{x : x \text{ is even number}\}$
  Simple sets like these seem to have their modes of presentations built into them. For instance, $6 \in b$ because b is the canonical presentation of the evens $\{0, 2, 4, 6, 8, \ldots\}$ and I know that 6 is among this collection by virtue of the presentation.

- But arbitrary subsets of the natural numbers don't necessarily have their modes of presentations built into them.

How much set theory can you get?

- Answer: probably more than you would expect, but less than you might like.
- In the associated *JSL* paper, we show that you can get all of *ZFC-P*, that is Zermelo-Fraenkel set theory without powerset.
- And if you like you can get some limited amount of iterations of powerset.
- In particular, one can get the well-founded extensions to be elementary equivalent to $(L_\beta, \in)$ wherein $\beta = (\omega_{\alpha+1})^L$.
- Here, the well-founded extensions are simply the extensions such that its transitive closure under the defined membership relation is well-founded.
- But, it's worth emphasizing, it turns out that (demonstrably) the well-founded extensions don't satisfy full powerset.

Outline, where we are at

- *I. Introduction*
- *II. Neutral Core of Church's Intensional Logic*
- *III. Formalized Version of Russell-Myhill Paradox*
- *IV. Predicative Response to Russell-Myhill*
- *V. The Problem of Many Non-Extensions*
- ***VI. Conclusions and Further Questions***

Summing Up

- The aim has been to set out a predicative response to the Russell-Myhill paradox of propositions, within the framework of Church's intensional logic.

- I have not defended the choice of Church's intensional logic as opposed to the many varieties of other intensional logics. I adopted it only because it seemed versatile enough to frame both the predicative response and the more traditional possible worlds semantics.

- Further, I haven't given any argument that this is the best solution or the right solution. Rather, the aim has merely been to set out the predicative response in a clear way.

- Presumably figuring out what the right solution is would go hand-in-hand with seeing whether the predicative response could do equally well at other tasks of intensional logic. . . . . .

Further Questions

- *First*: we want intensional logics to be able to interpret categorical grammar and to the provide a semantics for belief attributions. Not at all obvious that the framework of the predicative response can do this.

- *Second*: I mentioned that there are two traditional interpretations of Fregean sense, namely a "route to reference" and definite descriptions. There are formal models in which the interpretation of Fregean senses are either types of algorithms or types of definite descriptions. But I don't know if there's any difference between these two that is detectable in the object language itself.

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